PHYSICS 717 PROBLEM SET 12

due: Friday, May 1, 2009, at the beginning of lecture

Problems

1.: Compute the stress tensor for the weak field gravity waves in vacuum starting from the action derived in class:

$$S = \frac{M_p^2}{32\pi} \int d^4x \left\{ \frac{1}{2} \partial_\lambda h^{\mu\nu} \partial^\lambda h_{\mu\nu} - \frac{1}{2} \partial_\lambda h \partial^\lambda h - \partial_\lambda h^{\lambda\nu} \partial^\mu h_{\mu\nu} + \partial^\nu h \partial^\mu h_{\mu\nu} \right\}$$

in radiation gauge. (hint: Allow $\eta_{\mu\nu}$, the metric of the background on which $h_{\mu\nu}$ is propagating, to vary away from the flat metric and compute

$$T_{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta S}{\delta g^{\mu\nu}}$$

and evaluate the functional with $h_{\mu\nu}$ set to vacuum radiation gauge solution.)

answer:

Compute the variation:

$$\begin{aligned}
T_{\mu\nu} &= 2\frac{\delta S}{\delta g^{\mu\nu}} \\
&= \frac{M_p^2}{16\pi} \Big[\eta^{\alpha\beta} \partial_\lambda h_{\mu\beta} \partial^\lambda h_{\nu\alpha} + \frac{1}{2} \partial_\mu h^{\alpha\beta} \partial_\nu h_{\alpha\beta} \\
&\quad - \frac{1}{2} \partial_\mu h \partial_\nu h - \partial_\lambda h_{\mu\nu} \partial^\lambda h \\
&\quad - \partial_\mu h_{\nu\beta} \eta^{\beta\lambda} \partial^\gamma h_{\gamma\lambda} - \partial^\lambda h_{\lambda\mu} \partial^\gamma h_{\gamma\nu} - \partial_\lambda h^{\lambda\gamma} \partial_\mu h_{\nu\gamma} \\
&\quad + \partial^\lambda h \partial_\mu h_{\nu\lambda} + \partial_\mu h \partial^\lambda h_{\lambda\nu} + \partial^\lambda h_{\mu\nu} \partial^\alpha h_{\alpha\lambda} \Big] - \eta_{\mu\nu} \mathcal{L} \\
&= \frac{M_p^2}{32\pi} \partial_\mu h^{\alpha\beta} \partial_\nu h_{\alpha\beta}
\end{aligned}$$

which is what we aimed for.

2.: Using Noether's procedure with the $SO(3,1)$ Lie group parameterization matrix ω (in the notation of lecture 36) given by

$$\omega^{\mu\alpha} = \epsilon(\delta\,^\mu{}_i \delta^\alpha{}_j - \delta\,^\alpha{}_i \delta^\mu{}_j),$$

find the angular momentum density (angular momentum per unit spatial volume) for gravity waves in radiation gauge expressed in terms of $h_{\mu\nu}$. Use the answer to problem 1 to identify the orbital angular momentum part and the spin angular momentum part.

answer:

Under rotations

$$\begin{aligned}
\delta h^{\alpha\gamma} &= \frac{-i}{2} \omega^{\mu\nu} \left[(J_{\mu\nu})^\alpha{}_\beta h^{\beta\gamma} + (J_{\mu\nu})^\gamma{}_\beta h^{\alpha\beta} \right] \\
&= \frac{-i}{2} \omega^{\mu\nu} \left[\left[-i \left(\delta^\alpha{}_\mu g_{\nu\beta} - \delta^\alpha{}_\nu g_{\mu\beta} \right) \right] h^{\beta\gamma} + \left[-i \left(\delta^\gamma{}_\mu g_{\nu\beta} - \delta^\gamma{}_\nu g_{\mu\beta} \right) \right] h^{\alpha\beta} \right] \\
&= \frac{-1}{2} \left[\omega^{\alpha\nu} h_\nu{}^\gamma - \omega^{\mu\alpha} h_\mu{}^\gamma + \omega^{\gamma\nu} h^\alpha{}_\nu - \omega^{\mu\gamma} h^\alpha{}_\mu \right] \\
&= \frac{-1}{2} \left[-2\omega^{\mu\alpha} h_\mu{}^\gamma - 2\omega^{\mu\gamma} h^\alpha{}_\mu \right] \\
&= \omega^{\mu\alpha} h_\mu{}^\gamma + \omega^{\mu\gamma} h^\alpha{}_\mu
\end{aligned}$$

where

$$\omega^{\mu\alpha} = \delta\,^\mu{}_i \delta^\alpha{}_j - \delta\,^\alpha{}_i \delta^\mu{}_j$$

Hence, using the Noether charge formula

$$\epsilon J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_\alpha)} \left[\partial_\nu \psi_\alpha \delta x^\nu - \delta \psi_\alpha \right] - \mathcal{L} \delta x^\mu$$

and

$$\begin{aligned}\delta x^\nu &= \frac{-i}{2}\omega^{\alpha\beta}(J_{\alpha\beta})^\nu{}_\mu x^\mu \\ &= \frac{-i}{2}\omega^{\alpha\beta}\left[-i\left(\delta^\nu{}_\alpha g_{\beta\mu} - \delta^\nu{}_\beta g_{\alpha\mu}\right)\right]x^\mu \\ &= \frac{-1}{2}\left(\omega^{\nu\beta}x_\beta - \omega^{\beta\nu}x_\beta\right) \\ &= \omega^{\beta\nu}x_\beta\end{aligned}$$

the angular momentum density is

$$\begin{aligned}\epsilon J^\nu_{(ij)} &= \frac{\partial\mathcal{L}}{\partial\partial_\nu h^{\alpha\gamma}}\left[\partial_\lambda h^{\alpha\gamma}\delta x^\lambda - \delta h^{\alpha\gamma}\right] - \mathcal{L}\delta x^\nu \\ &= \frac{\partial\mathcal{L}}{\partial\partial_\nu h^{\alpha\gamma}}\left[\partial_\lambda h^{\alpha\gamma}\omega^{\beta\lambda}x_\beta - \left(\omega^{\mu\alpha}h_\mu^{\ \gamma} + \omega^{\mu\gamma}h^\alpha{}_\mu\right)\right] - \mathcal{L}\omega^{\beta\nu}x_\beta.\end{aligned}$$

where (ij) specifies $\omega^{\mu\beta}$ to choose the rotation generators: i.e. $\omega_{0\mu} = 0$. Evaluating the density, we evaluate 0th component using $\mathcal{L} = \frac{M_p^2}{64\pi}\partial_\lambda h^{\mu\nu}\partial^\lambda h_{\mu\nu}$:

$$\begin{aligned}\epsilon J^0_{(ij)} &= \frac{\partial\mathcal{L}}{\partial\partial_0 h^{\alpha\gamma}}\left[\partial_\lambda h^{\alpha\gamma}\omega^{\beta\lambda}x_\beta - \left(\omega^{\mu\alpha}h_\mu^{\ \gamma} + \omega^{\mu\gamma}h^\alpha{}_\mu\right)\right] - \mathcal{L}\omega^{\beta 0}x_\beta \\ &= \frac{M_p^2}{32\pi}\partial^0 h_{\alpha\gamma}\partial_\lambda h^{\alpha\gamma}\omega^{\beta\lambda}x_\beta - \frac{M_p^2}{32\pi}\partial^0 h_{\alpha\gamma}\left(\omega^{\mu\alpha}h_\mu^{\ \gamma} + \omega^{\mu\gamma}h^\alpha{}_\mu\right)\end{aligned}$$

where we used $\omega^{\beta 0} = 0$. Simplifying, we find

$$\begin{aligned}J^0_{(ij)} &= \frac{M_p^2}{32\pi}\left[x_i\partial^0 h_{\alpha\gamma}\partial_j h^{\alpha\gamma} - \partial^0 h_{\alpha\gamma}\partial_i h^{\alpha\gamma}x_j\right] \\ &\quad -\frac{M_p^2}{32\pi}\partial^0 h_{\alpha\gamma}\left(\left(\delta^{\ \mu}_{\ i}\delta^\alpha{}_j - \delta^{\ \alpha}_{\ i}\delta^\mu{}_j\right)h_\mu^{\ \gamma} + \left(\delta^{\ \mu}_{\ i}\delta^\gamma{}_j - \delta^{\ \gamma}_{\ i}\delta^\mu{}_j\right)h^\alpha{}_\mu\right) \\ &= x^iT^{0j} - x^jT^{0i} - \frac{M_p^2}{32\pi}\partial^0 h_{\alpha\gamma}\left(\delta^\alpha{}_j h_i^{\ \gamma} - \delta^{\ \alpha}_{\ i}h_j^{\ \gamma} + h^\alpha{}_i\delta^\gamma{}_j - \delta^{\ \gamma}_{\ i}h^\alpha{}_j\right) \\ &= x^iT^{0j} - x^jT^{0i} - \frac{M_p^2}{32\pi}\left(h_i^{\ \gamma}\partial^0 h_{j\gamma} - h_j^{\ \gamma}\partial^0 h_{i\gamma} + h^\alpha{}_i\partial^0 h_{\alpha j} - h^\alpha{}_j\partial^0 h_{\alpha i}\right) \\ &= x^iT^{0j} - x^jT^{0i} - \frac{M_p^2}{16\pi}\left(h_i^{\ \gamma}\partial^0 h_{j\gamma} - h_j^{\ \gamma}\partial^0 h_{i\gamma}\right)\end{aligned}$$

where we used the solution to problem 1. This is what we expect for the angular momentum density where the term in the parantheses is the spin component.

3.: For the following plane wave solution in radiation gauge

$$h_{\mu\nu} = A\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \cos(k_z(t-z)) & \sin(k_z(t-z)) & 0 \\ 0 & \sin(k_z(t-z)) & -\cos(k_z(t-z)) & 0 \\ 0 & 0 & 0 & 0\end{pmatrix}$$

compute $J^0_{(12)}$ and T^{0z}. Interpret the result, given that from a quantum perspective $T^{0z}/(\hbar k_z)$ counts the number of gravitons per unit volume.

answer:

We find from

$$\begin{aligned}T^{0j} &= -\frac{M_p^2}{32\pi}\partial_0 h^{\alpha\beta}\partial_j h_{\alpha\beta} \\ &= \frac{A^2k_z^2M_p^2}{16\pi}\end{aligned}$$

that the orbital angular momentum part is zero.

$$\begin{aligned}J^0_{(12)} &= -\frac{M_p^2}{16\pi}\left(h_1^{\ \gamma}\partial^0 h_{2\gamma} - h_2^{\ \gamma}\partial^0 h_{1\gamma}\right) \\ &= \frac{A^2k_zM_p^2}{8\pi}.\end{aligned}$$

Hence,

$$\frac{J^0_{(12)}}{T^{0j}/(\hbar k_z)} = 2\hbar$$

which says that the gravitons in this configuration carry an average spin of $2\hbar$.

4.: Explicitly check that

$$[\Sigma^{\alpha\beta}, \Sigma^{\gamma\delta}] = \eta^{\gamma\beta}\Sigma^{\alpha\delta} - \eta^{\gamma\alpha}\Sigma^{\beta\delta} + \eta^{\delta\beta}\Sigma^{\gamma\alpha} - \eta^{\delta\alpha}\Sigma^{\gamma\beta}$$

where

$$\Sigma^{\alpha\beta} = \frac{1}{4}\left[\gamma^\alpha, \gamma^\beta\right]$$

with the metric signature $(1, -1, -1, -1)$ and γ^α are those given in class.

answer:

In class, we had

$$\gamma^\mu \equiv \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$

Hence,

$$\begin{aligned}
\{\gamma^\mu, \gamma^\nu\} &= \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^\nu \\ \bar{\sigma}^\nu & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma^\nu \\ \bar{\sigma}^\nu & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \\
&= \begin{pmatrix} \{\sigma^\nu, \bar{\sigma}^\mu\} & 0 \\ 0 & \{\sigma^\nu, \bar{\sigma}^\mu\} \end{pmatrix} \\
&= 2\eta^{\mu\nu}
\end{aligned}$$

Hence, we find

$$\begin{aligned}
[\Sigma^{\alpha\beta}, \gamma^\rho] &= \frac{1}{4}[\left[\gamma^\alpha, \gamma^\beta\right], \gamma^\rho] \\
&= \frac{1}{4}[\gamma^\alpha\gamma^\beta - \gamma^\beta\gamma^\alpha, \gamma^\rho] \\
&= \frac{1}{4}[\gamma^\alpha\gamma^\beta + \gamma^\beta\gamma^\alpha - \gamma^\beta\gamma^\alpha - \gamma^\beta\gamma^\alpha, \gamma^\rho] \\
&= \frac{1}{4}[\{\gamma^\alpha, \gamma^\beta\} - 2\gamma^\beta\gamma^\alpha, \gamma^\rho] \\
&= \frac{1}{2}[\eta^{\alpha\beta} - \gamma^\beta\gamma^\alpha, \gamma^\rho] \\
&= -\frac{1}{2}(\gamma^\beta\gamma^\alpha\gamma^\rho - \gamma^\rho\gamma^\beta\gamma^\alpha) \\
&= -\frac{1}{2}(\gamma^\beta\gamma^\alpha\gamma^\rho - \{\gamma^\rho, \gamma^\beta\}\gamma^\alpha + \gamma^\beta\gamma^\rho\gamma^\alpha) \\
&= -\frac{1}{2}(\gamma^\beta 2\eta^{\alpha\rho} - 2\eta^{\rho\beta}\gamma^\alpha) \\
&= \eta^{\rho\beta}\gamma^\alpha - \gamma^\beta\eta^{\alpha\rho}.
\end{aligned}$$

Finally, we can then easily compute

$$\begin{aligned}
[\Sigma^{\alpha\beta}, \Sigma^{\gamma\delta}] &= \frac{1}{4}[\Sigma^{\alpha\beta}, \left[\gamma^\gamma, \gamma^\delta\right]] \\
&= \frac{1}{4}[\Sigma^{\alpha\beta}, \gamma^\gamma\gamma^\delta] - \frac{1}{4}[\Sigma^{\alpha\beta}, \gamma^\delta\gamma^\gamma] \\
&= \frac{1}{4}[\Sigma^{\alpha\beta}, \gamma^\gamma]\gamma^\delta + \frac{1}{4}\gamma^\gamma[\Sigma^{\alpha\beta}, \gamma^\delta] - \frac{1}{4}[\Sigma^{\alpha\beta}, \gamma^\delta]\gamma^\gamma - \frac{1}{4}\gamma^\delta[\Sigma^{\alpha\beta}, \gamma^\gamma] \\
&= \frac{1}{4}(\eta^{\gamma\beta}\gamma^\alpha - \gamma^\beta\eta^{\alpha\gamma})\gamma^\delta + \frac{1}{4}\gamma^\gamma(\eta^{\delta\beta}\gamma^\alpha - \gamma^\beta\eta^{\alpha\delta}) - \frac{1}{4}(\eta^{\delta\beta}\gamma^\alpha - \gamma^\beta\eta^{\alpha\delta})\gamma^\gamma - \frac{1}{4}\gamma^\delta(\eta^{\gamma\beta}\gamma^\alpha - \gamma^\beta\eta^{\alpha\gamma}) \\
&= \eta^{\gamma\beta}\Sigma^{\alpha\delta} - \eta^{\gamma\alpha}\Sigma^{\beta\delta} + \eta^{\delta\beta}\Sigma^{\gamma\alpha} - \eta^{\delta\alpha}\Sigma^{\gamma\beta}
\end{aligned}$$

completing the proof.

5.: Suppose the action governing an electron field is

$$S = \int d^4x\sqrt{g}\bar{\psi}\left[i\gamma^\nu D_\nu - m\right]\psi$$

where D_ν is the covariant derivative defined in class. Compute the stress energy tensor for the electron field by varying the tetrad. (Work with the metric signature (1,-1,-1,-1).)

answer:

As shown in lecture 40, we have

$$\begin{aligned} T_{cf} &= (e_\gamma)_f \left\{ i\bar{\psi}\gamma^\gamma D_c\psi + i\bar{\psi}\gamma^\mu (e_\mu)^a \frac{1}{2}\Sigma^{\gamma\epsilon}\nabla_c (e_\epsilon)_a \psi \right. \\ & \left. -i\nabla_b \left[\bar{\psi}\gamma^\mu (e_\mu)_c \frac{1}{2}\Sigma^{\delta\gamma}(e_\delta)^b\psi \right] \right\} - g_{fc}\left[\bar{\psi}\left[i\gamma^\nu D_\nu - m\right]\psi\right] \end{aligned}$$

where

$$D_c \equiv \partial_c + \frac{1}{2}\Sigma^{\alpha\beta}(e_\alpha)^b \nabla_b (e_\beta)_c$$

and

$$\Sigma^{\alpha\beta} = \frac{1}{4}[\gamma^\alpha, \gamma^\beta].$$