PHYSICS 717 PROBLEM SET 5

due: Monday, March 2, 2009, at the beginning of lecture

Problems

(1) Astronomical observations of the brightness of objects are measurements of the flux of radiation T^{0i} from the object at Earth. Assume there is no gravity (only special relativity applies).
   (a) Show that in the rest frame O of a star of constant luminosity L (total energy radiated per second), the stress-energy tensor of the radiation from the star at the event $(t, x, 0, 0)$ has components $T^{00} = T^{0x} = T^{x0} = T^{xx} = \frac{L}{4\pi x^2}$ if the star sits at the origin.
   (b) Let X be a null 4-vector which separates the events of emission and reception of the radiation. Show that $X = (x, x, 0, 0)$ in frame O for radiation observed at the event $(x, x, 0, 0)$. Show that the stress-energy tensor of part(a) has the frame-invariant form

$$T = \frac{L}{4\pi} \frac{X \otimes X}{(g_{ab}X^a U^b)^4}$$

   where U is the 4-velocity of the star which in the frame O is $(1, 0, 0, 0)$.
   (c) Let the Earth-bound observer $\bar{O}$, traveling with speed v away from the star in the x direction, measure the same radiation, again with the star on the $\bar{x}$ axis. Let $X = (R, R, 0, 0)$ in frame $\bar{O}$. Find R as a function of x. Express $T^{\bar{0}\bar{x}}$ in terms of R. In words, physically interpret the results in the limit $v \to 1$.

(2) Show that

$$\nabla_a j^a = 0$$

picks out the Maxwell's equation

$$\nabla^a \nabla_a A_b - R^d{}_b A_d = -j_b$$

instead of

$$\nabla^a \nabla_a A_b = -j_b$$

that one would obtain from the naive substitution rule.

(3) Completing the arguments of lecture 13, show that

$$T^{\alpha\beta}_{(EM)} \equiv F^\alpha{}_\gamma F^{\beta\gamma} - \frac{1}{4}\eta^{\alpha\beta} F_{\gamma\delta} F^{\gamma\delta}$$

satisfies

$$\partial_\alpha T^{\alpha\beta}_{(EM)} = 0$$

if $dF = 0$ and $J^\gamma = 0$ (no external source). Covariantizing this to curved spacetime, is variationally derived Einstein equations consistent with the variational definition of stress energy tensor and

$$S_{EM} = \frac{1}{4}\int d^4x \sqrt{-g} F_{ab} F^{ab}?$$

Why or why not?

(4) Compute the Einstein tensor component G_{00} for the metric

$$ds^2 = -e^{2\Phi(r)} dt^2 + e^{2\Lambda(r)} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

where $\Phi(r)$ and $\Lambda(r)$ are smooth functions of r.

(5) Suppose you are given an action for a complex scalar field Φ charged under electromagnetism as

$$S[g_{\mu\nu}, \Phi, A_\mu] = \int d^4x \sqrt{-g}\left\{ -(\partial_\mu \Phi^* + ieA_\mu \Phi^*)(\partial^\mu \Phi - ieA^\mu \Phi) + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \lambda(|\Phi|^2 - \frac{\sigma^2}{2})^2 \right\}$$

where $F = dA$ as usual and σ is a number. Compute the total stress energy tensor that couples to gravity. Is the stress tensor symmetric in $\mu \leftrightarrow \nu$? Why or why not?