

1)

The geodesic equation for  $\theta$  is given by

$$\frac{d^2\theta}{d\lambda^2} + \Gamma_{\alpha\beta}^2 \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = 0.$$

$$\Gamma_{\alpha\beta}^2 = \frac{1}{2} g^{2\lambda} (g_{\alpha\lambda,\beta} + g_{\beta\lambda,\alpha} - g_{\alpha\beta,\lambda})$$

In the Schwarzschild background

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} + r^2 d\Omega^2.$$

Hence

$$\Gamma_{\alpha\beta}^2 = \frac{1}{r} (\delta_{\alpha 2} \delta_{\beta 1} + \delta_{\alpha 1} \delta_{\beta 2} - \delta_{\alpha 3} \delta_{\beta 3} r \sin \theta \cos \theta)$$

With  $\theta = \pi/2$ ,

$$\Gamma_{\alpha\beta}^2 = \frac{1}{r} (\delta_{\alpha 2} \delta_{\beta 1} + \delta_{\alpha 1} \delta_{\beta 2})$$

However, since

$$\frac{d\theta}{d\lambda} = 0$$

with  $\theta = \pi/2$ , we have

$$\Gamma_{\alpha\beta}^2 \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = 0$$

and hence  $\theta = \pi/2$  satisfies the geodesic equation.

2)

Start from

$$\phi = \phi_0 + \frac{2M}{b} + \arcsin(by) - 2M \sqrt{\frac{1}{b^2} - y^2}.$$

Solving for the argument of the arcsin function, we have

$$by = \sin\left(\phi - \phi_0 - \frac{2M}{b} + 2M \sqrt{\frac{1}{b^2} - y^2}\right)$$

Let  $\delta \equiv \frac{M}{b}$  (which is a small quantity for physical cases of interest discussed in lecture) and  $z \equiv by$ , we have

$$z = \sin\left(\phi - \phi_0 - 2\delta + 2\sqrt{\delta^2 - \delta^2 z^2}\right).$$

Expanding to linear order in  $\delta$ , we find

$$z = \sin(\phi - \phi_0) - \cos(\phi - \phi_0) 2\delta (1 - \sqrt{1 - z^2}) + O(\delta^2).$$

Solving for  $z$ , we find

$$z = \sin(\phi - \phi_0) - 2\delta \cos(\phi - \phi_0) \pm 2\delta \cos^2(\phi - \phi_0)$$

Since  $z \rightarrow 0$  as  $\sin(\phi - \phi_0) \rightarrow 0$ , we can choose the positive sign:

$$z = \sin(\phi - \phi_0) - 2\delta \cos(\phi - \phi_0) [1 - \cos(\phi - \phi_0)].$$

Since

$$\begin{aligned}y &\equiv u(1 - Mu) \\by &= bu(1 - \frac{M}{b}bu) \\z &= \tilde{u}(1 - \delta\tilde{u})\end{aligned}$$

where  $\tilde{u} \equiv bu$ ,

$$-\delta\tilde{u}^2 + \tilde{u} - z = 0$$

Using quadratic formula this can be solved for

$$\tilde{u} = \frac{-1 + \sqrt{1 - 4\delta z}}{-2\delta}.$$

Expanding about  $\delta = 0$ , we have

$$\tilde{u} = z + \delta z^2 + O(\delta^2)$$

Substituting in for  $z$ , we have

$$\begin{aligned}\tilde{u} &= \sin(\phi - \phi_0) - 2\delta \cos(\phi - \phi_0)(1 - \cos(\phi - \phi_0)) + \delta \sin^2(\phi - \phi_0) + O(\delta^2) \\&= \sin(\phi - \phi_0) + \delta(1 - \cos(\phi - \phi_0))^2 + O(\delta^2)\end{aligned}$$

Hence, we have the desired result:

$$bu = \sin(\phi - \phi_0) + \frac{M}{b}[1 - \cos(\phi - \phi_0)]^2 + O([\frac{M}{b}]^2).$$

3)

since we know from lecture 21 that  $\delta\phi \approx 1.75$  arcsec for  $b \approx 6.95 \times 10^5$  km and since

$$\delta\phi \propto b^{-1},$$

we can scale this answer to obtain

$$b \approx 10^7 \text{ km.}$$

4)

From the Killing vector equation

$$g_{\alpha\beta,\mu}\xi^\mu + g_{\alpha\beta}\xi_{,\alpha}^\delta + g_{\alpha\delta}\xi_{,\beta}^\delta = 0,$$

we see that  $\xi^\mu = (1, 1, 0, 0)$  is a timelike Killing vector if

$$r - t > 4M/3.$$

Hence, we can check hypersurface orthogonality by using Frobenius's theorem: i.e. see if

$$\xi_{[\gamma}\xi_{\alpha,\beta]} = 0.$$

Since

$$\xi_\alpha = (1, \frac{4}{9}(\frac{9M}{2(r-t)})^{2/3}, 0, 0)$$

has only (0, 1) nonvanishing components depending on only (0, 1) coordinates, we see that the hypersurface orthogonality condition is automatically satisfied. Hence, this metric is static if  $r - t > 4M/3$ .

5)

a)

Note that in spherical coordinates, we have

$$F_{ab} = \begin{pmatrix} 0 & E^r & E^\theta & E^\phi \\ -E^r & 0 & B^\phi & -B^\theta \\ -E^\theta & -B^\phi & 0 & B^r \\ -E^\phi & B^\theta & -B^r & 0 \end{pmatrix}.$$

Hence, spherical symmetry means that  $E^\theta = E^\phi = B^\theta = B^\phi = 0$ . Since the form of the metric must be compatible with the static spherical symmetry condition, we can choose the basis (similar to lecture and discussed in Wald)

$$(e_0)_a = \sqrt{f}(dt)_a$$

$$(e_1)_a = \sqrt{h}(dr)_a$$

$$(e_2)_a = r(d\theta)_a$$

$$(e_3)_a = r \sin \theta (d\phi)_a.$$

We can therefore write the most general static spherically symmetric two-form  $F_{ab}$  in this basis as

$$F_{ab} = 2A(r)(e_0)_{[a}(e_1)_{b]} + 2B(r)(e_2)_{[a}(e_3)_{b]}.$$

**b)**

If we assume  $B(r) = 0$ , from part a), we can write

$$F = A(r)e_0 \wedge e_1.$$

Hence, the dual is given by

$$\begin{aligned} *F &= A(r)\epsilon^{01}_{cd}e^c \wedge e^d = -A(r)e^2 \wedge e^3 \\ &= -A(r)r^2 \sin \theta d\theta \wedge d\phi \end{aligned}$$

The source-free Maxwell equations is then

$$d^*F = 0 = -[A'(r)r^2 + 2rA(r)] \sin \theta d\theta \wedge d\phi = 0.$$

This implies

$$\frac{A'(r)}{A(r)} = \frac{-2}{r}$$

which can be easily integrated to give

$$A(r) = \frac{-q}{r^2}$$

**c)**

The stress energy tensor is given by

$$\begin{aligned} T_{ab} &= \frac{1}{4\pi} \{F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F_{de}F^{de}\} \\ &= \frac{1}{\pi} \{A^2(r)(e_0)_{[a}(e_1)_{c]}(e_0)_{[b}(e_1)_{e]}g^{ec} - \frac{1}{4}g_{ab}A^2(r)(e_0)_{[d}(e_1)_{e]}(e_0)_{[f}(e_1)_{g]}g^{fd}g^{ge}\} \end{aligned}$$

where we are using Wald's convention for charge normalization. The components are

$$\begin{aligned} T_{00} &= \frac{1}{\pi} \{A^2(r)(e_0)_{[0}(e_1)_{c]}(e_0)_{[0}(e_1)_{e]}g^{ec} - \frac{1}{4}g_{00}A^2(r)(e_0)_{[d}(e_1)_{e]}(e_0)_{[f}(e_1)_{g]}g^{fd}g^{ge}\} \\ &= \frac{1}{\pi} \left\{ \frac{1}{4}A^2(r)(e_0)_0(e_1)_c(e_0)_0(e_1)_e g^{ec} - \frac{1}{4}g_{00}A^2(r)(e_0)_{[d}(e_1)_{e]}(e_0)_{[f}(e_1)_{g]}g^{fd}g^{ge} \right\} \\ &= \frac{1}{\pi} \left\{ \frac{1}{4}A^2(r)(e_0)_0(e_0)_0\eta_{11} - \frac{1}{4}g_{00}A^2(r)\frac{1}{2}[\eta_{00}\eta_{11}] \right\} \\ &= \frac{1}{\pi} \left\{ \frac{1}{4}A^2(r)f - \frac{1}{8}fA^2(r) \right\} \\ &= \frac{A^2(r)}{8\pi}f = \frac{q^2}{8\pi r^4}f. \end{aligned}$$

$$\begin{aligned}
T_{11} &= \frac{1}{\pi} \{ A^2(r)(e_0)_{[1}(e_1)_{c]}(e_0)_{[1}(e_1)_{e]}g^{ec} - \frac{1}{4}g_{11}A^2(r)(e_0)_{[d}(e_1)_{e]}(e_0)^{[d}(e_1)^{e]} \} \\
&= \frac{1}{\pi} \{ \frac{1}{4}A^2(r)[(e_0)_c(e_1)_1(e_0)_e(e_1)_1g^{ec}] - \frac{1}{4}g_{11}\frac{1}{2}A^2(r)\eta_{00}\eta_{11} \} \\
&= \frac{1}{\pi} \{ \frac{1}{4}A^2(r)[-(e_1)_1(e_1)_1] + \frac{1}{8}g_{11}A^2(r) \} \\
&= \frac{1}{\pi} \{ -\frac{1}{4}A^2(r)h + \frac{1}{8}hA^2(r) \} \\
&= \frac{-1}{8\pi}A^2(r)h = \frac{-q^2}{8\pi r^4}h
\end{aligned}$$

$$\begin{aligned}
T_{22} &= \frac{1}{\pi} \{ A^2(r)(e_0)_{[2}(e_1)_{c]}(e_0)_{[2}(e_1)_{e]}g^{ec} - \frac{1}{8}g_{22}A^2(r) \} \\
&= \frac{-1}{8\pi}r^2A^2(r) = \frac{-q^2}{8\pi r^2}
\end{aligned}$$

From page 125 of Wald (there is an error in Wald's book which you can correct by comparing with lecture notes), one can write the 00 Einstein's equation as

$$(rh^2)^{-1}h' + r^{-2}(1 - h^{-1}) = \frac{q^2}{r^4}$$

This can be written as

$$\frac{1}{r^2} \frac{d}{dr} [r(1 - h^{-1})] = \frac{q^2}{r^4}$$

Then, just as in the Schwarzschild case, we have

$$h = \frac{1}{1 - \frac{2m(r)}{r}}$$

with

$$\begin{aligned}
m(r) &= \frac{4\pi q^2}{8\pi} \int dr' r'^2 \frac{1}{r'^4} + a \\
&= -\frac{q^2}{2} \frac{1}{r} + a
\end{aligned}$$

Since, we know this must reduce to  $M$  in the limit  $q \rightarrow 0$  (Schwarzschild), we have

$$m(r) = M - \frac{q^2}{2r}$$

This means that

$$h = \frac{1}{1 - \frac{2M}{r} - \frac{q^2}{r^2}}$$

The 11 Einstein gives

$$\frac{f'}{rfh} - \frac{(1 - \frac{1}{h})}{r^2} = \frac{-q^2}{r^4},$$

which yields

$$\int \frac{df}{f} = \int \frac{2M - \frac{2q^2}{r}}{q^2 + r(r - 2M)} dr.$$

The integral can be evaluated as

$$f = 1 - \frac{2M}{r} + \frac{q^2}{r^2}.$$

Hence, we have found the charged black hole solution to be

$$ds^2 = -(1 - \frac{2M}{r} + \frac{q^2}{r^2})dt^2 + \frac{dr^2}{1 - \frac{2M}{r} - \frac{q^2}{r^2}} + r^2 d\Omega^2.$$

6)

From the definition of frequency of electromagnetic waves, we have

$$w = -\frac{dS}{d\tau} = -U^\mu k_\mu$$

where  $S$  is the phase of the electromagnetic wave,  $k_\mu$  is the null geodesic, and  $U^\mu$  is the 4-velocity of the observer defining  $d\tau$ . The 4-velocity of the radially infalling observer is

$$U_i^\mu = \left( \frac{dt_i}{d\tau}, \frac{dr_i}{d\tau}, 0, 0 \right)$$

and the null vector of the photon geodesic is

$$k^\mu = \left( \frac{dt_\gamma}{d\lambda}, \frac{dr_\gamma}{d\lambda}, 0, 0 \right).$$

The explicit expressions for the components were worked out in lecture:

$$k_0 = -E_\gamma = \text{constant}$$

$$\frac{dr_\gamma}{d\lambda} = E_\gamma$$

$$\frac{dt_i}{d\tau} = \frac{E_i}{-g_{00}}$$

$$\frac{dr_i}{d\tau} = -\sqrt{E_i^2 - \left(1 - \frac{2M}{r_i}\right)}.$$

Hence, we find

$$\begin{aligned} U_i^\mu k_\mu &= \frac{-E_i E_\gamma}{1 - \frac{2M}{r_i}} - \frac{\sqrt{E_i^2 - \left(1 - \frac{2M}{r_i}\right)} E_\gamma}{1 - \frac{2M}{r_i}} \\ &= -\frac{E_i E_\gamma}{1 - \frac{2M}{r_i}} \left[ 1 + \sqrt{1 - \frac{1}{E_i^2} \left(1 - \frac{2M}{r_i}\right)} \right]. \end{aligned}$$

The observer at  $r = \infty$  sees

$$w_\infty = E_\infty E_\gamma \left[ 1 + \sqrt{1 - \frac{1}{E_\infty^2}} \right]$$

The observer at  $r \sim 2M$  sees frequency

$$w_{\text{particle}} \sim \frac{2E_{\text{particle}} E_\gamma}{1 - \frac{2M}{r}}$$

as long as

$$\frac{w_\infty}{w_{\text{particle}}} \sim \frac{E_\infty}{2E_{\text{particle}}} \left(1 - \frac{2M}{r}\right) \left[ 1 + \sqrt{1 - \frac{1}{E_\infty^2}} \right]. \quad (1)$$

Now, the trajectory of a radially infalling particle obtained by taking the ration of  $dr/d\tau$  and  $dt/d\tau$ :

$$\frac{dr}{dt} = -\left[1 - \frac{2M}{r}\right] \sqrt{1 - \frac{1}{E^2} \left(1 - \frac{2M}{r}\right)}$$

As  $r \rightarrow 2M$ , we have

$$\frac{dr}{dt} \approx -\left[\frac{r}{2M} - 1\right]$$

Integrating, we have

$$2M \ln\left(\frac{r_E}{2M} - 1\right) \sim -t_E$$

where we have emphasized that this time is the emission time by putting an  $E$  subscript. The signal reaches  $r$  by integrating along the photon geodesic:

$$\begin{aligned} \frac{dr_\gamma}{dt_\gamma} &= \frac{dr_\gamma/d\lambda}{dt_\gamma/d\lambda} \\ &= E_\gamma / (-E_\gamma g^{00}) \\ &= 1 - \frac{2M}{r_\gamma} \end{aligned}$$

giving

$$r_\gamma + 2M \ln(r_\gamma - 2M) = t_\gamma + C$$

or

$$t_\infty - t_E = r_\infty - r_E + 2M \ln \left[ \frac{\frac{r_\infty}{2M} - 1}{\frac{r_E}{2M} - 1} \right]$$

yielding

$$\begin{aligned} 2M \ln\left(\frac{r_E}{2M} - 1\right) &\sim -t_\infty + r_\infty - r_E + 2M \ln \left[ \frac{\frac{r_\infty}{2M} - 1}{\frac{r_E}{2M} - 1} \right] \\ 4M \ln\left(\frac{r_E}{2M} - 1\right) &\sim -t_\infty + r_\infty - 2M - \delta + 2M \ln \left[ \frac{r_\infty}{2M} - 1 \right]. \end{aligned}$$

Note that as  $r_E \rightarrow 2M$ ,  $-t_\infty$  dominates over  $r_\infty$  in a divergent manner. Hence

$$r_E \sim 2M \left\{ 1 + \exp\left(-\frac{t_\infty}{4M}\right) \right\}$$

Putting this into Eq. (1), we find

$$\frac{w_\infty}{w_{\text{particle}}} \sim \frac{E_\infty}{2E_{\text{particle}}} \exp\left(-\frac{t_\infty}{4M}\right) \left[ 1 + \sqrt{1 - \frac{1}{E_\infty^2}} \right]$$

or

$$k = 4M.$$