PHYSICS 717 PROBLEM SET 8

due: Monday, March 30, 2009, at the beginning of lecture

Problems

1.: Problem 4 on page 158 of Wald.

answer:

a):

Let ξ^α be a timelike Killing vector. Since ξ^a is timelike, we can choose the time coordinate to be generated by ξ^a. In that case, a stationary observer has a 4-velocity proportional to ξ^a:

$$u^a = N\xi^a$$

where N is a normalization constant. By the definition of timelikeness, we have

$$u^a u_a = N^2 \xi^a \xi_a = -1,$$

yielding

$$N = \sqrt{-1/(\xi^a \xi_a)} \equiv \frac{1}{V}$$

.

Now, the acceleration of a stationary observer is given as

$$a^b = u^a \nabla_a u^b$$

which yields

$$\begin{aligned} a^b &= \frac{\xi^a}{V}\nabla_a[\frac{\xi^b}{V}] = \frac{\xi^a}{V}\frac{\nabla_a \xi^b}{V} - \frac{\xi^b \xi^a}{V}\frac{\nabla_a V}{V^2} \\ &= \frac{\xi^a}{V}\frac{\nabla_a \xi^b}{V} + \frac{\xi^b \xi^a}{V^2}\frac{\xi^c \nabla_a \xi_c}{V^2} \end{aligned}$$

Since $\xi_{(a;b)} = 0$, the last term vanishes:

$$\frac{\xi^b \xi^a}{V^2}\frac{\xi^c \nabla_a \xi_c}{V^2} = 2\frac{\xi^b \xi^a}{V^2}\frac{\xi^c \nabla_{(a}\xi_{c)}}{V^2} = 0.$$

Since

$$\nabla_a \xi_b = -\nabla_b \xi_a,$$

the remaining term gives

$$\begin{aligned} a^b &= -\frac{\xi_a}{V}\frac{\nabla^b \xi^a}{V} \\ &= \frac{1}{2}\frac{\nabla^b(-\xi_a \xi^a)}{V^2} \\ &= \frac{1}{2}\nabla^b \ln V^2 = \nabla^b \ln V. \end{aligned}$$

b):

Consider the local change in energy

$$\Delta E = q^\mu \nabla_\mu E$$

where q^μ is a spacelike separation representing the displacement of the rope. Since

$$E = -m\xi^a u_a = -m\xi^a \xi_a \frac{1}{V} = mV,$$

we have

$$\Delta E = m q^\mu \nabla_\mu V.$$

Finally, since part a) gave us

$$\nabla_\mu V = V a_\mu,$$

we find

$$\begin{aligned}\Delta E &= mVq^\mu a_\mu \\ &\equiv Vq^\mu F_\mu.\end{aligned}$$

Hence, imposing energy conservation

$$\Delta E(\infty) - \Delta E = 0,$$

we find

$$q^\mu(\infty)F_\mu(\infty)V(\infty) = q^\mu F_\mu V.$$

Note that $V(\infty) = 1$ since we have asymptotic flatness.

Now, we need to choose q^μ. We choose it to be in the direction of the force which is spacelike, and we choose it's length to be independent of whether it is at ∞ or not. Hence, we find

$$q^\mu = \frac{F^\mu}{\sqrt{F^\mu F_\mu}}.$$

Hence, we arrive at

$$F(\infty) = FV$$

as desired.

2.: Simple coordinate exercises:

a): Show that Schwarzschild curvature singularity at $r = 0$ is a spacelike hypersurface.

answer:

As proven as a class exercise during lecture, in the conformal coordinates defining the Penrose diagram, $r = 0$ maps to $\psi = \pi/2$. Since any given point A on $\psi = \pi/2$ surface has a past null causal cone eminating at $\pi/4$ angles from it, every neighboring point of A on $r = 0$ surface is spacelike separated.

b): Show that for Schwarzschild $r < 2M$, surfaces of constant Schwarzschild t are straight lines in Kruskal coordinates and that surfaces of constant r are hyperbolas in Kruskal coordinates.

answer:

For $r < 2M$, we have

$$\frac{T}{X} = \tanh\frac{t}{4M}.$$

Hence, constant t surfaces are straight lines. For constant r, we have

$$T^2 - X^2 = (1 - \frac{r}{2M})e^{r/(2M)}$$

which shows that constant r surfaces are hyperbolas.

3.: Referring to the construction of a Kruskal wormhole in Euclidean coordinate embedding, suppose you have the solutions $\{r(t), z(t)\}$ to the equations (where $z(t)$ represents all branches if it involves square roots)

$$(\frac{dr}{dt})^2 + (\frac{dz}{dt})^2 = \frac{2M}{r} - 1$$

$$\sqrt{1 - \frac{r}{2M}}e^{r/(4M)}\cosh\left[t/(4M)\right] = T_1$$

inside the horizon. Find the functions (of t and ϕ) in the Euclidean vector

$$(x(t,\phi), y(t,\phi), z(t,\phi))$$

that can be used to plot the wormhole. [i.e. express the answer including $r(t)$ and $z(t)$]

answer:

Since the radius of the cylindrical embedding is $r(t)$, we have

$$x = r(t)\cos\phi$$

$$y = r(t)\sin\phi$$

$$z = z(t)$$

4.: Write a single algebraic equation (not differential) determining the minimum Euclidean radius of the Krukal wormhole throat constructed in class as a function of T_1 denoted in lecture.

answer:

As discussed in lecture or from problem 2, we have

$$\lim_{X\to 0}(T_1^2 - X^2) = \lim_{r\to r_{\text{min}}} \left[(1-\frac{r}{2M})e^{r/(2M)}\right]$$

yielding

$$T_1 = \sqrt{1-\frac{r_{\text{min}}}{2M}}e^{r_{\text{min}}/(4M)}.$$

5.: A particle of mass $m > 0$ falls radially towards the horizon of a Schwarzschild black hole of mass M. The geodesic it follows has $E = 0.95$.

a): Find the proper time required to reach $r = 2M$ from $r = 3M$.

answer:

We have from lecture

$$\int d\tau = -\int \frac{dr}{\sqrt{E^2 - V^2}}$$

$$\Delta\tau = -\int_{r_i}^{r_f} \frac{dr}{\sqrt{\frac{2M}{r} - (1-E^2)}}$$

Let

$$\frac{2M}{r} = [1-E^2]\sec^2\theta.$$

This yields

$$\Delta\tau = \frac{4M}{[1-E^2]^{3/2}}\int_{\arccos\sqrt{\frac{[1-E^2]}{2M/r_i}}}^{\arccos\sqrt{\frac{[1-E^2]}{2M/r_f}}} d\theta \cos^2\theta$$

$$= \frac{2M}{[1-E^2]^{3/2}}\left[\theta + \frac{1}{2}\sin 2\theta\right]_{\arccos\sqrt{\frac{[1-E^2]}{2M/r_i}}}^{\arccos\sqrt{\frac{[1-E^2]}{2M/r_f}}}.$$

For $r_i = 3M$ to $r_f = 2M$, we find

$$\Delta\tau \approx 1.2M$$

b): Find the proper time required to reach $r = 0$ from $r = 2M$.

answer:

With the formula in part a), we find

$$\Delta\tau \approx 1.4M$$

c): In Schwarzschild coordinate basis, find its 4-velocity components at $r = 2.001$M.

answer:

The 4-velocity components in Schwarzschild coordinate basis is

$$u^\mu = (\frac{dt}{d\tau}, \frac{dr}{d\tau}, 0, 0)$$

which as worked out in lecture is

$$u^\mu = (\frac{E}{1-\frac{2M}{r}}, -\sqrt{\frac{2M}{r} - (1-E^2)}, 0, 0).$$

Evaluating for the given parameters at $r = 2.001M$, we find

$$u^\mu = (1901, -0.9497, 0, 0)$$

6.: In Schwarzschild geometry, compute the $\hat{\phi}$ direction tidal acceleration on a pair of closeby $\hat{\phi}$ separated particles that are in geodesic motion, if they are released from rest at a coordinate location $(r > 2M, \theta = \pi/2, \phi)$? (i.e. Use geodesic deviation.) Suppose the tolerance of human body is a tidal acceleartion of 500 m/s^2. How massive must the black hole be for human beings to survive the $\hat{\phi}$ tidal acc as the body crosses the horizon?

answer:

The geodesic deviation equation derived in lecture is

$$a^a = -R^a_{cbd} X^b T^c T^d$$

where we required

$$X^a T_a = 0$$

and we can normalize

$$T^a T_a = -1$$

corresponding to normalizing the acceleration with respect to proper time. Since released from rest, we have

$$\begin{aligned} T^a &= \left(\frac{E}{1-\frac{2M}{r}}, -\sqrt{\frac{2M}{r}-(1-E^2)}, 0, 0\right) \\ &= \left(\frac{1}{\sqrt{1-\frac{2M}{r}}}, 0, 0, 0\right) \end{aligned}$$

$$X^a = (0, 0, \xi^\phi, 0).$$

We thus find

$$a^\phi = \frac{d\xi^\phi}{d\tau^2} = \frac{-M}{r^3}\xi^\phi.$$

Putting in ordinary units, we have

$$\frac{d\xi^\phi}{d\tau^2} = -G_N \frac{M}{r^3}\xi^\phi.$$

If the human body length scale is L, we have as the human being crosses the horizon $(\xi^\phi)^2 r^2 = L^2$. Hence, we find

$$\begin{aligned} r\frac{d\xi^\phi}{d\tau^2}\Big|_{r=2M} &= -G_N \frac{M}{(2G_N M/c^2)^3} L \\ &= -\frac{c^6 L}{8G_N^2 M^2} \end{aligned}$$

Setting this equal to $-500m/s^2$, we find

$$\begin{aligned} M &= \sqrt{\frac{c^6 L}{8G_N^2(500m/s^2)}} \\ &\approx 6\times 10^{33} \text{ kg} \\ &\approx 3\times 10^3 \text{solar masses} \end{aligned}$$

where we have taken the typical human body length scale to be about $L \approx$1 m. Hence, the black hole mass should be much larger than the mass of the sun.