Physics 717 Problem set 9

April 10, 2009

due Monday, April 6, 2009, at the beginning of lecture

Problems

1. Complete the argument in lecture in showing that

$$\frac{dP}{dr} = -(\rho + P)\left\{\frac{\frac{4\pi}{3}r^3\rho + 4\pi r^3 P}{r^2\left[1 - \frac{2m(r)}{r}\right]}\right\}$$

with $m(r)$ given as in lecture for the the equation of state of ρ being a constant independent of P can be integrated to give

$$\frac{\rho + 3P}{\rho + P} = \frac{\sqrt{1 - \frac{2m(r)}{r}}}{\sqrt{1 - \frac{2M}{R}}}.$$

Be sure to use the boundary conditions discussed in class.

answer

Putting in

$$m(r) = \frac{4\pi}{3}r^3\rho,$$

we find

$$\int \frac{dP}{(\rho + P)(\rho + 3P)} = -\frac{4\pi}{3}\int \frac{rdr}{\left[1 - \frac{8\pi}{3}r^2\rho\right]}.$$

Since ρ is a constant, one can compute the integral

$$\frac{1}{2\rho}\ln\left[\frac{P + \rho/3}{P + \rho}\right] = \frac{1}{4\rho}\ln\left[8\pi r^2\rho - 3\right] + C$$

where C is a constant. Taking the log of both sides, we obtain

$$\frac{\rho + 3P}{\rho + P} = A\sqrt{1 - \frac{2m(r)}{r}}$$

where A is another constant. Using the boundary condition that $P(R) = 0$ and $m(R) = M$, we find

$$1 = A\sqrt{1 - \frac{2M}{r}},$$

we find

$$A = \frac{1}{\sqrt{1 - \frac{2M}{r}}}.$$

Hence, we arrive at the result

$$\frac{\rho + 3P}{\rho + P} = \frac{\sqrt{1 - \frac{2m(r)}{r}}}{\sqrt{1 - \frac{2M}{r}}}.$$

2. From the result of problem 1, we have

$$P(r) = \frac{\rho\left[\sqrt{1-\frac{2Mr^2}{R^3}} - \sqrt{1-\frac{2M}{R}}\right]}{3\sqrt{1-\frac{2M}{R}} - \sqrt{1-\frac{2Mr^2}{R^3}}}$$

Integrate the $\Phi'(r)$ disucssed in class to obtain

$$\exp\left[2\Phi(r)\right] = (1-\frac{2M}{R})\left[\frac{\rho}{\rho+P(r)}\right]^2$$

using the boundary conditions discussed in class.

answer

The equation governing Φ is

$$(\rho+P)\frac{d\Phi}{dr} = -\frac{dP}{dr}.$$

Integrating, we find

$$\Phi = -\ln(\rho+P) + C.$$

Using the result of problem 1, we have

$$\Phi = -\ln\left(\rho + \left\{\frac{\rho\left[\sqrt{1-\frac{2Mr^2}{R^3}} - \sqrt{1-\frac{2M}{R}}\right]}{3\sqrt{1-\frac{2M}{R}} - \sqrt{1-\frac{2Mr^2}{R^3}}}\right\}\right) + C$$

Set the boundary condition of matching Schwarzschild at $r = R$:

$$\Phi(R) = \frac{1}{2}\ln\left(1-\frac{2M}{R}\right).$$

Hence, we find

$$C = \frac{1}{2}\ln\left\{\rho^2\left(1-\frac{2M}{R}\right)\right\}$$

and

$$\exp\left[2\Phi(r)\right] = \left(1-\frac{2M}{R}\right)\left[\frac{\rho}{\rho+P}\right]^2$$

3. Verify the statement about spherically symmetric pressureless dust with the boundary conditions given in class collapsing in finite proper time to reach the singularity: i.e. Integrate

$$(\frac{\dot{a}}{a})^2 + \frac{k}{a^2} = \frac{8\pi}{3}\rho_0(\frac{a_0}{a})^3.$$

In what sense is $a = 0$ singular?

answer

Solving for $\dot{a}$, we can express the solution for collapse to $a_f = 0$ as

$$\int_1^{a_f/a_0} \frac{d(a/a_0)}{\sqrt{\frac{8\pi}{3}\rho_0(\frac{a_0}{a}) - \frac{k}{a_0^2}}} = -[t_f - t_0].$$

Recalling that by definition of the boundary condition used in class,

$$\frac{k}{a_0^2} = \frac{8\pi}{3}\rho_0,$$

We can then rewrite the integral as

$$\frac{1}{\sqrt{\frac{8\pi}{3}\rho_0}} \int_0^1 \frac{dx}{\sqrt{\frac{1}{x}-1}} = t_f - t_0$$

Carrying out the integral, we find

$$\frac{\sqrt{3\pi}}{\sqrt{2\rho_0}} \frac{1}{4} = t_f - t_0$$

which is clearly finite.

Note that the trace of Einstein eq is

$$-R = 8\pi T = -8\pi\rho$$

which diverges at $a \to 0$ since $\rho \propto a^{-3}$. Hence, there is a coordinate invariant curvature singularity there.

4. If a photon is emittted from the surface of a spherical star of uniform density ρ, show that the observed redshift must obey the constraint $z < 2$.

answer

Since the bound on M/R is

$$\frac{M}{R} \leq \frac{4}{9},$$

and since the redshift formula is

$$\frac{1}{1+z} = \frac{\sqrt{-g_{00}(r=R)}}{\sqrt{-g_{00}(r=\infty)}} = e^{\Phi(R)-\Phi(\infty)}$$

which gives

$$\begin{aligned} \frac{1}{1+z_{max}} &== \sqrt{1-\frac{2M}{R}} \\ &= \sqrt{1-\frac{8}{9}} = \frac{1}{3} \end{aligned}$$

or equivalently

$$z_{max} = 2$$

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