Physics 721 official solutions

Homework 3

① $L = -m\sqrt{1-(\dot{r}^2+r^2\dot{\phi}^2)} + \frac{e^2}{r}$

$S = \int dt\, L(r,\phi,\dot{r},\dot{\phi})$

a) $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0$

$\Rightarrow$ Since L does not depend on ϕ, $K = \frac{\partial L}{\partial \dot{\phi}}$ is conserved.

$$H = \dot{\phi}\frac{\partial L}{\partial \dot{\phi}} + \dot{r}\frac{\partial L}{\partial \dot{r}} - L$$

$$= \dot{q}^i \pi^i - L \quad \text{where} \quad \pi^i = \frac{\partial L}{\partial \dot{q}^i} \quad q^i \in \{\phi, r\}$$

$$\frac{d}{dt}H = \ddot{q}^i \pi^i + \dot{q}^i \dot{\pi}^i - \frac{d}{dt}L$$

$$\frac{d}{dt}L = \ddot{q}^i \frac{\partial L}{\partial \dot{q}^i} + \dot{q}^i \frac{\partial L}{\partial q^i} + \cancelto{0}{\frac{\partial L}{\partial t}}$$

$$\therefore \quad \frac{d}{dt}H = \cancel{\ddot{q}^i \pi^i} + \cancel{\dot{q}^i \dot{\pi}^i} - \left(\cancel{\ddot{q}^i \frac{\partial L}{\partial \dot{q}^i}} + \cancel{\dot{q}^i \frac{\partial L}{\partial q^i}}\right)$$

$$= \boxed{0}$$

Hence we see as long as $\frac{\partial L}{\partial t} = 0$, H is always conserved.

b) $$H = \dot{\phi}\frac{\partial L}{\partial \dot{\phi}} + \dot{r}\frac{\partial L}{\partial \dot{r}} - L = \dot{\phi}\left[\frac{m \cancel{2}\cancel{\frac{1}{2}} r^2 \dot{\phi}}{\sqrt{1-(\dot{r}^2+r^2\dot{\phi}^2)}}\right] + \dot{r}\left[\frac{m \cancel{2}\cancel{\frac{1}{2}} \dot{r}}{\sqrt{1-(\dot{r}^2+r^2\dot{\phi}^2)}}\right] - L$$

$$= \frac{m r^2\dot{\phi}^2 + m\dot{r}^2 + m\left(1-(\dot{r}^2+r^2\dot{\phi}^2)\right)}{\sqrt{1-(\dot{r}^2+r^2\dot{\phi}^2)}} - \frac{e^2}{r}$$

$$= \frac{m}{\sqrt{1-(\dot{r}^2+r^2\dot{\phi}^2)}} - \frac{e^2}{r}$$

$$K = \frac{\partial L}{\partial \dot\phi} = \frac{m r^2 \dot\phi}{\sqrt{1-(\dot r^2 + r^2\dot\phi^2)}}$$

Solving for $\dot\phi$, we find $\dot\phi = \frac{K\sqrt{1-\dot r^2}}{r\sqrt{K^2+m^2r^2}}$

Hence,

$$H = \frac{K}{r^2\dot\phi} - \frac{e^2}{r}$$

$$\boxed{\therefore\ H = \frac{1}{r}\frac{\sqrt{K^2+m^2r^2}}{\sqrt{1-\dot r^2}} - \frac{e^2}{r}}$$

1st order in $\dot r$ as requested.

c) Use Euler–Lagrange or Hamilton's eq. (both approaches below)

Euler – Lagrange:

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot r}\right] - \frac{\partial L}{\partial r} = 0$$

$$\frac{d}{dt}\left[\frac{m\dot r}{\sqrt{1-(\dot r^2+r^2\dot\phi^2)}}\right] + \frac{e^2}{r^2} + \frac{1}{2}\frac{m(-2r\dot\phi^2)}{\left[1-(\dot r^2+r^2\dot\phi^2)\right]^{1/2}} = 0$$

When $\dot r = 0$, $\dot\phi = \text{const}$

Hence,

$$\frac{e^2}{R^2} = \frac{mR\dot\phi^2}{(1-R^2\dot\phi^2)^{1/2}}$$

where we used $r = R = \text{const}$. Defining $\dot\phi = \frac{2\pi}{T}$, we find

$$\boxed{\frac{T^2}{(2\pi)^2}e^2 = \frac{mR^3}{\left(1-R^2\left(\frac{2\pi}{T}\right)^2\right)^{1/2}}}$$

In the limit that $R^2\dot\phi^2 \to 0$, we recover the usual $T^2 \propto R^3$.

alternate approaches:

Hamiltonian approach

Using $p_r \equiv \frac{\partial L}{\partial \dot{r}} = \frac{m\dot{r}}{\sqrt{1-(\dot{r}^2+r^2\dot{\phi}^2)}}$

we find by eliminating $\dot{\phi}$ and $\dot{r}$ in favor of K and p_r

$$H = \sqrt{m^2+p_r^2+\frac{K^2}{r^2}} - \frac{e^2}{r}$$

Hamiltonian equations of motion

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{\sqrt{m^2+p_r^2+\frac{K^2}{r^2}}} = 0 \quad \text{for } r = \overset{\text{const.}}{R} \text{ (circular)}$$

$$\dot{p}_r = -\frac{\partial H}{\partial r} = -\left[\frac{-\frac{K^2}{r^3}}{\sqrt{m^2+p_r^2+\frac{K^2}{r^2}}} + \frac{e^2}{r^2}\right] = 0 \quad \text{for } r = R$$

$$\dot{\phi} = \frac{\partial H}{\partial K} = \frac{\frac{K}{r^2}}{\sqrt{m^2+p_r^2+\frac{K^2}{r^2}}} = \text{const} = \frac{2\pi}{T} \quad \text{for } r = R$$

$$\dot{K} = -\frac{\partial H}{\partial \phi} = 0$$

$$\therefore \quad -\frac{2\pi K}{TR} + \frac{e^2}{R^2} = 0$$

$$\therefore \quad \frac{-2\pi}{TR}\,\frac{mR^2\left(\frac{2\pi}{T}\right)}{\sqrt{1-R^2\left(\frac{2\pi}{T}\right)^2}} + \frac{e^2}{R^2} = 0$$

$$\boxed{\therefore \quad \frac{(2\pi)^2}{e^2T^2} = \frac{\sqrt{1-R^2\left(\frac{2\pi}{T}\right)^2}}{R^3\, m}}$$

as before

2 Jackson 12.14 $\quad \mathcal{L} = -\frac{1}{8\pi}\partial_\alpha A_\beta \partial^\alpha A^\beta - J_\alpha A^\alpha \quad$ ($c = 1$ kept 4π)

a) $$\partial_\alpha \frac{\partial \mathcal{L}}{\partial(\partial_\alpha A_\beta)} - \frac{\partial \mathcal{L}}{\partial A_\beta} = 0$$

$$-\frac{1}{4\pi}\partial_\alpha \partial^\alpha A^\beta - - J^\beta = 0$$

$$\boxed{\Box A^\beta = 4\pi J^\beta}$$

equivalent to maxwell eq. if the following holds:

$$\boxed{\begin{aligned} \vec{B} &= \vec{\nabla} \times \vec{A} \\ \vec{E} &= -\vec{\nabla} A^0 - \partial_0 \vec{A} \\ \partial_\alpha A^\alpha &= 0 \end{aligned}}$$

b) (12.25) $$\mathcal{L}_g = -\frac{1}{16\pi} F_{\alpha\beta}F^{\alpha\beta} - J_\alpha A^\alpha$$
$$= -\frac{1}{8\pi}\left(\partial_\alpha A_\beta \partial^\alpha A^\beta - \partial_\beta A^\alpha \partial_\alpha A^\beta \right) - J_\alpha A^\alpha$$

$$\partial_\alpha \left[(\partial_\beta A^\alpha) A^\beta \right] = \partial_\beta A^\alpha \partial_\alpha A^\beta + (\partial_\alpha \partial_\beta A^\alpha) A^\beta$$

$$\boxed{\begin{aligned} &\therefore \quad \mathcal{L} = \mathcal{L}_g - \underbrace{\frac{\partial_\alpha [\partial_\beta A^\alpha A^\beta]}{8\pi}}_{\text{total divergence}} \\ &\text{if} \quad \partial_\alpha A^\alpha = 0 \end{aligned}}$$

This added total divergence does not affect the action or the equation of motion as long as $(\partial_\beta A^\alpha) A^\beta n_\alpha = 0$ where n_α is the normal to the 3-volume bounding the spacetime.