Physics 721 Homework 8 Official Solutions

1 Jackson 4.5

a) $\vec{F} = \int \vec{E}\rho\, d^3x$

$$\vec{E} = -\vec{\nabla}\,\Phi^{(0)}$$

$$= -\hat{e}^i\partial_i\left[\Phi^{(0)}(0) + x^j\partial_j\Phi^{(0)}(0) + \frac{x^j x^k}{2}\partial_j\partial_k\Phi^{(0)}(0) + \frac{x^j x^k x^l}{6}\partial_j\partial_k\partial_l\Phi^{(0)}(0) + \dots\right]$$

constant

$$= -\hat{e}^i\partial_i\Phi^{(0)} + -\hat{e}^i x^k\partial_i\partial_k\Phi^{(0)}(0) + -\frac{\hat{e}^l}{2}x^k x^l\partial_i\partial_k\partial_l\Phi^{(0)}(0) + \dots$$

where we have used the convention that repeated indices are summed.

$$\therefore \vec{F} = q\,\vec{E}^{(0)}(0) + \int(\vec{x}\cdot\vec{\nabla})\vec{E}^{(0)}(0)\,\rho\, d^3x + \frac{1}{2}\int x^k x^l\partial_k\partial_l\vec{E}^{(0)}(0)\rho\, d^3x + \dots$$

$$\vec{\nabla}\left[\vec{p}\cdot\vec{E}^{(0)}(\vec{x})\right]\Big|_{x=0} = (\vec{p}\cdot\vec{\nabla})\vec{E}^{(0)}(0) \quad \text{since } \vec{p} \text{ is a constant.}$$

where $\vec{p} = \int d^3x'\,\vec{x}'\rho(\vec{x}')$

$$Q_{ij} = \int(3x_i'x_j' - r'^2\delta_{ij})\rho(\vec{x}')\,d^3x'$$

Note $\frac{1}{2}\int d^3x\; x^k x^l\partial_k\partial_l\vec{E}^{(0)}(0)\,\rho\, d^3x$

$$= -\frac{1}{2}\int d^3x\,\hat{e}^i x^k x^l\partial_k\partial_l\partial_i\Phi^{(0)}(0)\,\rho\, d^3x$$

$$= -\frac{1}{2}\left[\int d^3x\;\hat{e}^i\rho(x)\,x^k x^l\right]\partial_k\partial_l\partial_i\Phi^{(0)}(0)$$

$$= -\frac{\hat{e}^i}{2}\left[\frac{1}{3}Q_{kl} + \frac{\delta_{kl}}{3}\int x'^2\rho(\vec{x}')\,d^3x'\right]\partial_k\partial_l\partial_i\Phi^{(0)}(0)$$

since $\Phi^{(0)}$ is an $\boxed{\text{external}}$ potential,

$$\vec{\nabla}\cdot\vec{E}^{(0)} = -\nabla^2\Phi^{(0)} = 0.$$

$$\therefore \frac{1}{2}\int d^3x\; x^k x^l\partial_k\partial_l\vec{E}^{(0)}(0)\,\rho\, d^3x = -\frac{\hat{e}^i}{6}Q_{kl}\partial_k\partial_l\partial_i\Phi^{(0)}(0)$$

$$= +\frac{1}{6}\vec{\nabla}\left(Q_{kl}\,\partial_k E_l^{(0)}(\vec{x})\right)\Big|_{x=0}$$

$$\boxed{\therefore\; \vec{F} = q\,\vec{E}^{(0)}(0) + \vec{\nabla}\left[\vec{p}\cdot\vec{E}^{(0)}(\vec{x})\right]\Big|_{x=0} + \frac{1}{6}\vec{\nabla}\left(Q_{kl}\,\partial_k E_l^{(0)}(\vec{x})\right)\Big|_{x=0} + \dots}$$

note symmetric.