Lecture 3 (1/24/05)

Relativsitic Electrodynamics (continued)

(finish last lecture)

9. 4-vector nature of p^μ

10. Relativistic tensors

11. Field Strength Tensor and Maxwell Equations (alternate to 11.9, pg. 553)

We can rewrite Maxwell's equations suggestively using

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix}$$

$$\partial_\mu F^{\mu\nu} = J^\nu$$

$$\partial_\mu \tilde{F}^{\mu\nu} = 0$$

where

$$J^\mu = (\rho, \vec{J})$$

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta} \quad \epsilon^{0123} \equiv 1,\ \ \epsilon^{1023} = -1,\ \ \epsilon^{1203} = 1,\ \text{etc.}$$

e.g.

$$\partial_0 F^{01} + \partial_i F^{i1} = -\partial_0 E^1 + \partial_2 B^3 - \partial_3 B^2 = J^1.$$

which agrees with

$$\nabla \times \vec{B} = \partial_t \vec{E} + \vec{J}$$

since

$$\partial_2 B^3 - \partial_3 B^2 = (\nabla \times \vec{B})^1.$$

We cannot conclude that this is a tensor until we define the Lorentz transformation properties of J^μ.

For point particles, we have

$$\vec{J}(\vec{x}, t) \equiv \sum_n e_n \delta^{(3)}(\vec{x} - \vec{x}_n(t)) \frac{d\vec{x}_n(t)}{dt}$$

$$J^0 = \rho(\vec{x}, t) \equiv \sum_n e_n \delta^{(3)}(\vec{x} - \vec{x}_n(t))$$

Therefore, defining $x_n^0(t) \equiv t$, we can write

$$\begin{aligned} J^\alpha &= \sum_n e_n \delta^{(3)}(\vec{x} - \vec{x}_n(t)) \frac{dx_n^\alpha(t)}{dt} \\ &= \int dt' \sum_n e_n \delta^{(4)}(x - x_n(t')) \frac{dx_n^\alpha(t')}{dt'} \\ &= \sum_n \int d\tau_n e_n \delta^{(4)}(x - x_n(t'(\tau_n))) \frac{dx_n^\alpha(t'(\tau_n))}{d\tau_n} \end{aligned}$$

Under Lorentz transformations, we have

$$\delta^{(4)}(\Lambda^\gamma{}_\beta[x^\beta - x_n^\beta(t'(\tau))]) = \frac{\delta^{(4)}(x^\beta - x_n^\beta(t'(\tau)))}{\det\Lambda^\gamma{}_\beta} = \delta^{(4)}(x^\beta - x_n^\beta(t'(\tau))).$$

Thus the delta function is a scalar and the current transforms like a 4-vector. Therefore, our Maxwell equations written in terms of F is a tensor equation. The tensor $F^{\alpha\beta}$ is called the field strength tensor.