Physics 801 Problem Set 1 Suggested Solutions

1.

The derivation of the Friedman equations follows directly from computing the Christoffel symbols in the $\{t, r, \theta, \phi\}$ coordinates and the necessary derivatives. Given the form of the metric, it is easy to show

$$\begin{aligned} \Gamma^{\lambda}_{\mu\nu} &= \frac{1}{2} g^{\lambda\lambda}[g_{\lambda\lambda,1}(\delta_{\mu\lambda}\delta_{\nu1} + \delta_{\nu\lambda}\delta_{\mu1}) + g_{\lambda\lambda,0}(\delta_{\mu\lambda}\delta_{\nu0} + \delta_{\nu\lambda}\delta_{\mu0}) + \\ &\quad \delta_{\lambda3} g_{33,2}(\delta_{\nu2}\delta_{\mu3} + \delta_{\mu2}\delta_{\nu3}) - g_{\mu\nu,\lambda}] \end{aligned}$$

where here we are not summing over any repeated indices. The ten $R_{\mu\nu}$ can thus easily be computed as

$$R_{\mu\nu} = \begin{pmatrix} -3\frac{\ddot{a}}{a} & 0 & 0 & 0 \\ 0 & \frac{a^2}{1-kr^2}[\frac{\ddot{a}}{a} + 2(\frac{\dot{a}}{a})^2 + 2\frac{K}{a^2}] & 0 & 0 \\ 0 & 0 & a^2r^2[\frac{\ddot{a}}{a} + 2(\frac{\dot{a}}{a})^2 + 2\frac{K}{a^2}] & 0 \\ 0 & 0 & 0 & a^2r^2\sin^2\theta[\frac{\ddot{a}}{a} + 2(\frac{\dot{a}}{a})^2 + 2\frac{K}{a^2}] \end{pmatrix}.$$

Hence, the Ricci Scalar is

$$R = -6[\frac{\ddot{a}}{a} + (\frac{\dot{a}}{a})^2 + \frac{K}{a^2}]$$

Since ${T^0}_0 = \rho(t)$, we find

$$R_{00} - \frac{1}{2}g_{00}R = -3\frac{\ddot{a}}{a} + 3[\frac{\ddot{a}}{a} + (\frac{\dot{a}}{a})^2 + \frac{K}{a^2}] = \frac{8\pi}{M_{pl}^2}T_{00} = \frac{8\pi}{M_{pl}^2}\rho$$

or equivalently

$$(\frac{\dot{a}}{a})^2 + \frac{K}{a^2} = \frac{8\pi}{3M_{pl}^2}\rho$$

which is the Friedmann equation.

2.

Start with the metric on the 3-sphere

$$ds^2 = \sum_{i=1}^{4}(dx^i)^2.$$

Since the 3-sphere is defined by

$$\sum_{i=1}^{4}(x^i)^2 = \alpha^2,$$

we have

$$(x^4)^2 = \alpha^2 - r^2$$

where

$$r^2 \equiv \sum_{i=1}^{3}(x^i)^2.$$

Hence,

$$dx^4 = \frac{rdr}{\sqrt{\alpha^2 - r^2}}$$

and

$$(dx^1)^2 + (dx^2)^2 + (dx^3)^2 = dr^2 + r^2d\Omega^2.$$

Hence, we arrive at

$$\begin{aligned} ds^2 &= dr^2(1 + \frac{r^2}{\alpha^2 - r^2}) + r^2d\Omega^2 \\ &= dr^2(\frac{1}{1 - \frac{r^2}{\alpha^2}}) + r^2d\Omega^2. \end{aligned}$$

3.

Taking the functional variation, we have

$$\begin{aligned}\delta S &= \int d^4x\sqrt{g}[\partial_{[\alpha}\chi_{\mu\nu]}\partial_{[\beta}\chi_{\lambda\kappa]}\delta g^{\alpha\beta}g^{\lambda\mu}g^{\kappa\nu} \\ &\quad \partial_{[\alpha}\chi_{\mu\nu]}\partial_{[\beta}\chi_{\lambda\kappa]}g^{\alpha\beta}\delta g^{\lambda\mu}g^{\kappa\nu} + \partial_{[\alpha}\chi_{\mu\nu]}\partial_{[\beta}\chi_{\lambda\kappa]}g^{\alpha\beta}g^{\lambda\mu}\delta g^{\kappa\nu} \\ &\quad +m^2\chi_{\mu\nu}\chi_{\alpha\beta}\delta g^{\mu\alpha}g^{\nu\beta} + m^2\chi_{\mu\nu}\chi_{\alpha\beta}g^{\mu\alpha}\delta g^{\nu\beta} - \frac{1}{2}g_{\mu\nu}\delta g^{\mu\nu}\{\partial_{[\alpha}\chi_{\mu\nu]}\partial_{[\beta}\chi_{\lambda\kappa]}g^{\alpha\beta}g^{\lambda\mu}g^{\kappa\nu} + m^2\chi_{\mu\nu}\chi^{\mu\nu}\}].\end{aligned}$$

Hence, we find

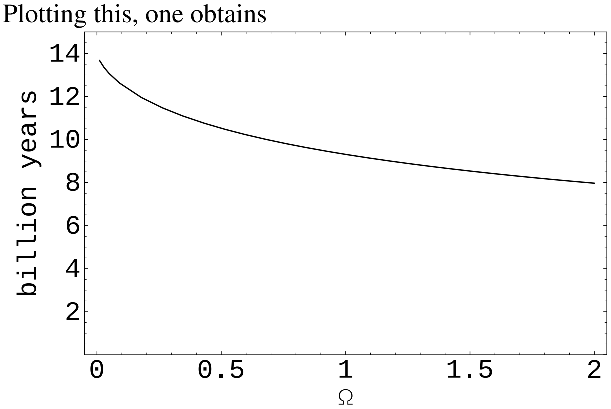
$$\begin{aligned}T_{\mu\nu}(x) &= \frac{2}{\sqrt{g}}\frac{\delta S}{\delta g^{\mu\nu}(x)} \\ &= 2[\partial_{[\mu}\chi_{\theta\phi]}\partial_{[\nu}\chi_{\lambda\kappa]}g^{\lambda\theta}g^{\kappa\phi} + \partial_{[\alpha}\chi_{\nu\gamma]}\partial_{[\beta}\chi_{\mu\kappa]}g^{\alpha\beta}g^{\kappa\gamma} + \partial_{[\alpha}\chi_{\gamma\nu]}\partial_{[\beta}\chi_{\lambda\mu]}g^{\alpha\beta}g^{\lambda\gamma} \\ &\quad +m^2\chi_{\mu\gamma}\chi_{\nu\beta}g^{\gamma\beta} + m^2\chi_{\lambda\mu}\chi_{\alpha\nu}g^{\lambda\alpha}] - g_{\mu\nu}\{\partial_{[\alpha}\chi_{\mu\nu]}\partial_{[\beta}\chi_{\lambda\kappa]}g^{\alpha\beta}g^{\lambda\mu}g^{\kappa\nu} + m^2\chi_{\mu\nu}\chi^{\mu\nu}\} \\ &= 2[3\partial_{[\mu}\chi_{\theta\phi]}\partial_{[\nu}\chi_{\lambda\kappa]}g^{\lambda\theta}g^{\kappa\phi} + 2m^2\chi_{\mu\gamma}\chi_{\nu\beta}g^{\gamma\beta}] \\ &\quad -g_{\mu\nu}\{\partial_{[\alpha}\chi_{\gamma\theta]}\partial_{[\beta}\chi_{\lambda\kappa]}g^{\alpha\beta}g^{\lambda\gamma}g^{\kappa\theta} + m^2\chi_{\alpha\beta}\chi^{\alpha\beta}\}\end{aligned}$$

4.

The age of the universe for a matter dominated universe is given by

$$t(\Omega_0) = \begin{cases} H_0^{-1}\frac{\Omega_0}{2(\Omega_0-1)^{3/2}}\left(\text{acos}(\frac{2}{\Omega_0}-1) - \frac{2}{\Omega_0}\sqrt{\Omega_0-1}\right) & \Omega_0 \geq 1 \\ H_0^{-1}\frac{\Omega_0}{2(1-\Omega_0)^{3/2}}\left(-\text{acosh}(\frac{2}{\Omega_0}-1) + \frac{2}{\Omega_0}\sqrt{1-\Omega_0}\right) & \Omega_0 < 1 \end{cases}.$$

Plotting this, one obtains



Most of the age of the current universe has been accumulated in the matter dominated regime. It is important to note that the time scale of cosmology is thus billions of years and this fact is not extremely sensitive to the amount of matter in the universe.