Homework 4 (due: 11/1/07) solutions:

1. Dodelson pg. 258 #8

answer

With $R = 0$, the damping scale is

$$\frac{1}{k_D^2} = c\int_0^{\eta}\frac{d\eta'}{6n_e\sigma_T a}$$

where $c \sim \mathcal{O}(1)$. Change the integration variable to a:

$$\begin{aligned}\frac{1}{k_D^2} &= c\int_0^{\eta}\frac{d\eta'}{6n_e\sigma_T a}\\ &= c\int_0^{a(\eta)}\frac{da}{6n_e\sigma_T a\dot{a}}\\ &= c\int_0^{a(\eta)}\frac{da}{6n_e\sigma_T a^3 H}\end{aligned}$$

where H is the Hubble expansion rate. We can then use the assumption that only matter and radiation exists to write

$$\begin{aligned}H &\approx \sqrt{\frac{8\pi}{3M_{pl}^2}(\rho_m + \rho_r)}\\ &= \sqrt{\frac{8\pi}{3M_{pl}^2}(\rho_{m(eq)}[\frac{a_{eq}}{a}]^3 + \rho_{r(eq)}[\frac{a_{eq}}{a}]^4)}\end{aligned}$$

where $\rho_{m(eq)}$ and $\rho_{r(eq)}$ correspond to matter and radiation energy densities at the time of matter-radiation equality. If we define $y \equiv a/a_{eq}$, we find

$$H = \sqrt{\frac{8\pi}{3M_{pl}^2}\rho_{m(eq)}}\frac{1}{y^{3/2}}\sqrt{1+\frac{1}{y}}$$

Putting this back into the integral gives

$$\frac{1}{k_D^2} = \frac{c}{a_{eq}^2}\int_0^{a(\eta)/a_{eq}}\frac{y^{3/2}dy}{6n_{e(eq)}\sigma_T\sqrt{\frac{8\pi}{3M_{pl}^2}\rho_{m(eq)}}\sqrt{1+\frac{1}{y}}}.$$

Integral can be simplified as

$$\int_0^{y_f}\frac{y^{3/2}dy}{\sqrt{1+\frac{1}{y}}} = \frac{2}{5}y_f^{5/2}\left\{5\sqrt{1+\frac{1}{y_f}} - \frac{20}{3}(1+\frac{1}{y_f})^{3/2} + \frac{8}{3}[(1+\frac{1}{y_f})^{5/2} - 1/y_f^{5/2}]\right\}$$

where $y_f \equiv a(\eta)/a_{eq}$. Hence, we find

$$\frac{1}{k_D^2} = \frac{c}{a_{eq}^2 15 n_{e(eq)}\sigma_T\sqrt{\frac{8\pi}{3M_{pl}^2}\rho_{c0}\Omega_m\frac{\rho_{m(eq)}}{\rho_{m0}}}}y_f^{5/2}f_D(y_f)$$

where

$$f_D(y_f) \equiv 5\sqrt{1+\frac{1}{y_f}} - \frac{20}{3}(1+\frac{1}{y_f})^{3/2} + \frac{8}{3}[(1+\frac{1}{y_f})^{5/2} - \frac{1}{y_f^{5/2}}]$$

Using Eq. (8.41)

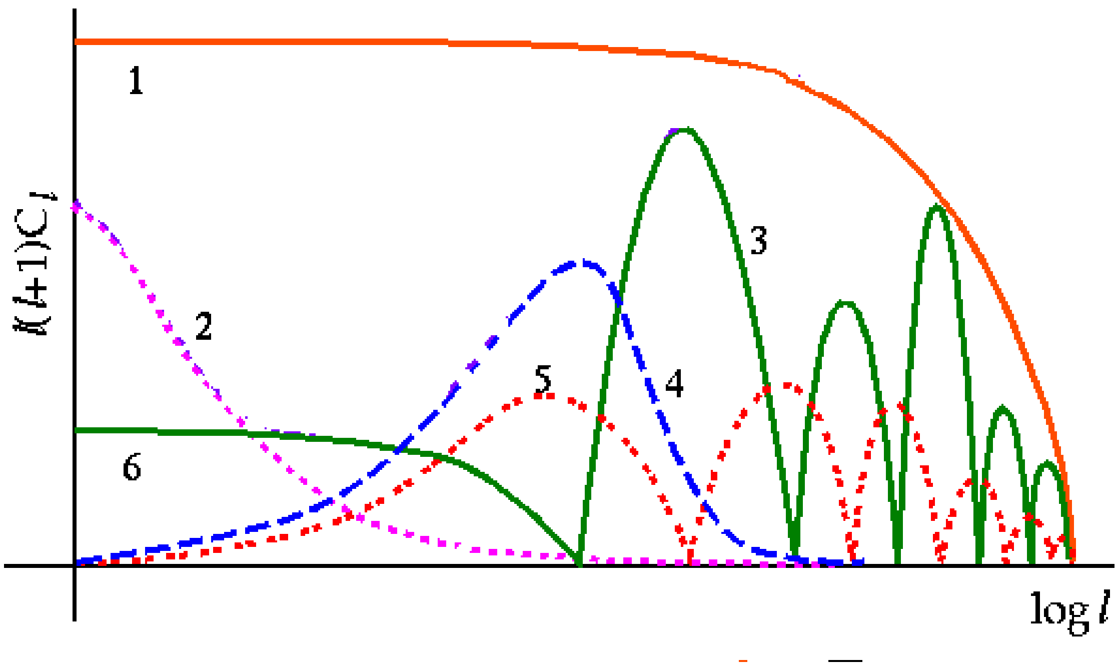
$$n_e\sigma_T a = 2.3\times10^{-5}\mathrm{Mpc}^{-1}\Omega_b h^2 a^{-2}(1-\frac{Y_p}{2})$$

we obtain the numerical value of

$$\frac{1}{k_D^2} = \frac{8.5 \times 10^6 c}{(\Omega_b h^2)\sqrt{\Omega_m h^2}} \left(1 - \frac{Y_p}{2}\right)^{-1} a^{5/2} f_D(a/a_{\rm eq}) {\rm Mpc}^2$$

where we have set $a_0 = 1$.

2. The figure below explicitly shows the various contributions to (or effects on) the temperature anisotropy spectrum in terms of $l(l+1)C_l$. Using the formulae discussed in class, justify (i.e. identify and discuss) each of the curves contributing to (or effecting) the spectrum. For example, curve labeled 2 corresponds to the envelope of the late time integrated Sachs-Wolfe effect while curve labeled 1 corresponds to a damping envelope (you still need to provide formulae for these).



answer

The C_l can be obtained from the formula

$$C_l = \frac{2}{\pi} \int \frac{dk k^2}{V} |\Theta_l(k)|^2$$

where V is the volume. Let's look at the different contribution to or features of Θ_l.

Curve 1):

This is the damping envelope due to Thompson scattering diffusion:

$$e^{-k^2/k_D^2}$$

where k_D was computed in problem 1. Since the approximate conversion between length scales is

$$l \sim k\eta_0 \sim k\frac{2}{H_0}$$

the damping should start to occur at

$$l_D \sim k_D \frac{2}{H_0} \sim (0.2 \text{ Mpc}^{-1}) 2 (6000 \text{Mpc}) \sim \mathcal{O}(10^3).$$

where we have used $a \sim 10^{-3}$, $\Omega_b \approx 0.05$, $\Omega_m \approx 0.25$, and $h \approx 0.7$. This is consistent with figure 8.8 of Dodelson.

Curve 2):

This is coming from late time integrated Sachs-Wolfe effect due to the fact that the universe becomes dark energy dominated at late times:

$$\Theta_l \sim \int_0^{\eta_0} d\eta e^{-\tau}[\dot{\Psi} - \dot{\Phi}] j_l[k(\eta_0 - \eta)] \tag{1}$$

where when the universe becomes cosmological constant dominated, the potential derivative $\dot{\Phi} \approx -\dot{\Psi}$ does not vanish. Consider the potential evolution given by

$$k^2 \Phi = \frac{4\pi}{M_{pl}^2} a^2 [\rho_{DM}\delta + \rho_r \delta_r].$$

Expressing this in terms of the total energy density $\rho_{tot} = H^2 3M_{pl}^2/(8\pi)$, we have

$$\begin{aligned} \Phi &= \frac{3H^2}{2} \frac{8\pi}{3M_{pl}^2 H^2} \frac{a^2}{k^2} [\rho_{DM}\delta + \rho_r \delta_r] \\ &= \frac{3H^2}{2} \frac{a^2}{k^2} [\frac{\rho_{DM}}{\rho_{DM} + \rho_\Lambda + \rho_r}\delta + \frac{\rho_r}{\rho_{DM} + \rho_\Lambda + \rho_r}\delta_r]. \end{aligned}$$

Neglecting the δ_r constribution, we have

$$\Phi \propto \Phi_p(\vec{k}) \frac{3H^2}{2} \frac{a^2}{k^2} [\frac{1}{1 + \frac{\Omega_\Lambda}{\Omega_{DM}}(\frac{a}{a_0})^3 + \frac{\Omega_r}{\Omega_{DM}}(\frac{a_0}{a})}] D_1(a)$$

where

$$D_1(a) = \frac{5\Omega_m}{2} \frac{H(a)}{H_0} \int_0^a \frac{da'}{a_0} \left(\frac{a_0 H_0}{a' H(a')} \right)^3$$

for modes that have entered the horizon. The growth function for a constant equation of state w is given by

$$\frac{D_1(a)}{a} \approx \frac{5\Omega_m}{2} \left(\frac{1}{\frac{5}{2} + \frac{9}{2}w} \right) a^{3w}$$

and the Hubble expansion rate can be written as

$$H \approx H_0 a^{-3(1+w)/2}$$

where we have set $a_0 = 1$. Hence, we find

$$\Phi \propto \Phi_p(\vec{k}) \frac{15}{4} \frac{H_0^2}{k^2} [\frac{1}{1 + \frac{\Omega_\Lambda}{\Omega_{DM}} a^3 + \frac{\Omega_r}{\Omega_{DM}} a^{-1}}] \left(\frac{\Omega_m}{\frac{5}{2} + \frac{9}{2}w} \right). \tag{2}$$

Hence, we see that when the dark energy (here assumed to be the cosmological constant) starts to dominate $\Phi \propto a^{-3}$ for modes which have entered the horizon. For modes that are outside of the horizon, the potential Φ remains constant and therefore $\dot{\Phi}$ is negligible. This leads to an ISW boost on the horizon scale at the time of Ω_Λ domination through the $-\dot{\Phi}$ in Eq. (1). Note that $\dot{\Phi}$ increases as dark energy becomes more dominant. The horizon scale approximately corresponds to longest wavelengths. [On shorter wavelengths, we see that H_0^2/k^2 suppresses $\dot{\Phi}$ in equation Eq. (2).]

Curve 3):

This is the Sachs-Wolfe contribution from

$$\Theta_l \sim \int_0^{\eta} d\eta g(\eta)[\Theta_0 + \Psi] j_l[k(\eta_0 - \eta)] \tag{3}$$

where the $\Theta_0 + \Psi$ oscillate as function of time. The salient feature of this photon-baryon fluid oscillation pattern is the enhancement of the odd peaks (if we count the peak at $l \approx 200$ as the first peak) relative to the even peaks. In the tight coupling regime, we have the fluid equation

$$(\frac{d^2}{d\eta^2} + \frac{\dot{R}}{1+R}\frac{d}{d\eta} + k^2 c_s^2)(\Theta_0 + \Phi) = \frac{k^2}{3}[3c_s^2\Phi - \Psi]$$

In a time interval very short compared to the time variation of Ψ, this equation tells us that there is a "constant" solution

$$\Theta_0 + \Phi \approx \Phi - \frac{1}{3c_s^2}\Psi = \Phi - (1+R)\Psi \approx -2\Psi(0) + R\Phi(0)$$

which will act as an offset for the oscillatory part of the solution. More explicitly, since the solution for Θ_0 will be of the form

$$\Theta_0 + \Phi \approx -2\Psi(0) + R\Phi(0) + [\Theta_0(0) - \Phi(0)(1+R)]\cos(\int_0^\eta w d\eta) + ...$$

Now, since the Sachs-Wolfe effect of Eq. (3) comes from $\Theta_0 + \Psi$, we find

$$\Theta_0 + \Psi \approx \Theta_0 + \Phi - 2\Phi \approx R\Phi(0) + [\Theta_0(0) - \Phi(0)(1+R)]\cos(\int_0^\eta w d\eta) + ...$$

which means that C_l will inherit the l dependence from the time dependence of

$$|\Theta_0 + \Psi|^2 = R\Phi(0)^2 + [\Theta_0(0) - \Phi(0)(1+R)]^2\cos^2(\int_0^\eta w d\eta) + 2R\Phi(0)[\Theta_0(0) - \Phi(0)(1+R)]\cos(\int_0^\eta w d\eta) + ...$$

Since $\Theta_0(0) + \Psi(0) = \Theta_0(0) - \Phi(0)$ and $\Phi(0)$ have the opposite signs (since we have learned in class that on long wavelengths $\Theta_0(0) \approx \frac{2}{3}\Phi(0)$), we see that every odd peak (where the first odd peak is defined to be the one associated with the observed peak at $l \sim 200$) is enhanced relative to the even peaks.

Curve 4):

This corresponds to the early time integrated-Sachs-Wolfe effect. Consider Eq. (2) at the time of last scattering surface

$$\begin{aligned} \Phi &\propto \Phi_p(\vec{k})\frac{15}{4}\frac{H_0^2}{k^2}[\frac{1}{1+\frac{\Omega_r}{\Omega_{DM}}a^{-1}}]\left(\frac{\Omega_m}{\frac{5}{2}+\frac{9}{2}w}\right) \\ &\approx \Phi_p(\vec{k})\frac{15}{4}\frac{H_0^2}{k^2}[1 - \frac{\Omega_r}{\Omega_{DM}}a^{-1}]\left(\frac{\Omega_m}{\frac{5}{2}+\frac{9}{2}w}\right) \end{aligned}$$

which boosts the potential at the horizon scale at the earliest possible time when densities start to grow, namely, the time of matter-radiation equality. Hence, we expect there to be an enhancement at

$$\frac{a_{eq}}{a_0}\frac{k}{a_{eq}} \sim \frac{a_{eq}}{a_0}H_0(\frac{a_0}{a_{eq}})^{3/2}$$

which maps to

$$l_{\text{early ISW}} \sim 10^2.$$

Curve 5):

This curve comes from the Doppler term

$$\begin{aligned} \Theta_l &\sim \int_0^{\eta_0} d\eta g(\eta)\frac{iv_b(k,\eta)}{k}\frac{d}{d\eta}j_l[k(\eta_0 - \eta)] \\ &\sim -\int_0^{\eta_0} d\eta g(\eta) iv_b(k,\eta) j_l[k(\eta_0 - \eta - \frac{\pi}{2})]. \end{aligned} \quad (4)$$

Now, since

$$v_b \sim -3i\Theta_1 \approx -\sqrt{3}i[\Theta_0(0) + \Phi(0)]\sin[k\int_0^\eta d\eta c_s] + ...$$

we have compared to the Doppler part (Eq. (3)) similar magnitudes, but out of phase by $\pi/2$.

Curve 6):

This is the Sachs-Wolfe contribution where the modes are outside of the horizon at the last scattering surface. The Sachs-Wolfe is then given by

$$(\Theta_0 + \Psi)(k, \eta_*) = \frac{1}{3}\Psi(k, \eta_*) = \frac{-1}{3}\Phi(k, \eta_*) = \frac{-1}{6}\delta(\eta_*)$$

which means that overdensities correspond to observed cold spots.

3. Green's function probelm:

Consider the differential equation

$$y'' + p(x)y' + q(x)y = 0$$

on the interval $a \le x \le b$. Suppose we know two solutions $y_1(x)$ and $y_2(x)$ such that

$$y_1(a) = 0 \qquad y_2(a) \neq 0$$

$$y_1(b) \neq 0 \qquad y_2(b) = 0$$

Give the solution of the equation

$$y'' + p(x)y' + q(x)y = f(x)$$

which obeys the conditions $y(a) = y(b) = 0$ in the form

$$y(x) = \int_a^b G(x, x')f(x')dx'$$

where $G(x, x')$ is the Green's function which involves only the solutions y_1 and y_2 and assumes different functional forms for $x' < x$ and $x' > x$. Illustrate by solving

$$y''(x) + k^2 y(x) = f(x)$$

$$y(a) = y(b) = 0$$

[hint: Since $G(x, x')$ satisfies

$$\partial_x^2 G(x, x') + p(x)\partial_x G(x, x') + q(x)G(x, x') = \delta(x - x')$$

by construction, integrate this equation to obtain conditions for $G(x, x')$.]

answer

For $x < x'$, we have

$$G(x, x') \propto y_1(x)$$

and for $x > x'$, we have

$$G(x, x') \propto y_2(x).$$

Hence, we can write the general solution as

$$G(x, x') = C_1(x')y_1(x)\theta(x' - x) + C_2(x')y_2(x)\theta(x - x').$$

Integrating the differential equation, $G(x, x')$ must satisfy the boundary condition

$$\partial_x G(x, x')|_{x'-\epsilon}^{x'+\epsilon} + \int_{x'-\epsilon}^{x'+\epsilon} dx p(x)\partial_x G(x, x') = 1.$$

$$\partial_x G(x, x') = C_1(x')\partial_x y_1(x)\theta(x' - x) + C_2(x')\partial_x y_2(x)\theta(x - x') - C_1(x')y_1(x)\delta(x' - x) + C_2(x')y_2(x)\delta(x - x').$$

Hence, we obtain

$$C_2(x')\partial_x y_2(x') - C_1(x')\partial_x y_1(x') + p(x')[C_2(x')y_2(x') - C_1(x')y_1(x')] = 1 \tag{5}$$

Apply continuity at $x = x'$. Hence, we obtain

$$C_1(x')y_1(x') = C_2(x')y_2(x')$$

and from Eq. (5) that

$$C_2(x')\partial_x y_2(x') - C_1(x')\partial_x y_1(x') = 1.$$

Solving for $C_1(x')$ and $C_2(x')$, we find

$$\begin{aligned} C_1 &= \frac{y_2(x')}{y_1(x')\partial_x y_2(x') - y_2(x')\partial_x y_1(x')} \\ C_2 &= \frac{y_1(x')}{y_1(x')\partial_x y_2(x') - y_2(x')\partial_x y_1(x')} \end{aligned}$$

Hence, we find

$$G(x, x') = \frac{y_2(x')y_1(x)}{y_1(x')\partial_x y_2(x') - y_2(x')\partial_x y_1(x')}\theta(x' - x) + \frac{y_1(x')y_2(x)}{y_1(x')\partial_x y_2(x') - y_2(x')\partial_x y_1(x')}\theta(x - x').$$

The solution using this Green's

$$\begin{aligned} y(x) &= \int_x^b \frac{y_2(x')y_1(x)}{y_1(x')\partial_x y_2(x') - y_2(x')\partial_x y_1(x')} f(x')dx' + \int_a^x \frac{y_1(x')y_2(x)}{y_1(x')\partial_x y_2(x') - y_2(x')\partial_x y_1(x')} f(x')dx' && (6) \\ &= \int_a^b \frac{y_2(x')y_1(x)}{y_1(x')\partial_x y_2(x') - y_2(x')\partial_x y_1(x')} f(x')dx' - \int_a^x \frac{y_2(x')y_1(x)}{y_1(x')\partial_x y_2(x') - y_2(x')\partial_x y_1(x')} f(x')dx' && (7) \\ &+ \int_a^x \frac{y_1(x')y_2(x)}{y_1(x')\partial_x y_2(x') - y_2(x')\partial_x y_1(x')} f(x')dx' && (8) \\ &= \gamma y_1(x) + \int_a^x \frac{y_1(x')y_2(x) - y_2(x')y_1(x)}{y_1(x')\partial_x y_2(x') - y_2(x')\partial_x y_1(x')} f(x')dx' && (9) \end{aligned}$$

where

$$\gamma = \int_a^b \frac{y_2(x')}{y_1(x')\partial_x y_2(x') - y_2(x')\partial_x y_1(x')} f(x')dx'$$

is just a number. Eq. (9) is the form of the solution that was used in class.

To illustrate, let's solve

$$y''(x) + k^2 y(x) = f(x)$$

$$y(a) = y(b) = 0.$$

The homogeneous solutions are

$$y_1(x) = \sin(k[x - a]) \qquad y_2(x) = \sin(k[x - b]).$$

TheWronskian for this is

$$\begin{aligned} y_1(x')\partial_x y_2(x') - y_2(x')\partial_x y_1(x') &= k \sin(k[x - a]) \cos(k[x - b]) - k \sin(k[x - b]) \cos(k[x - a]) \\ &= k \sin(k[b - a]) \end{aligned}$$

The Green's function solution is

$$\begin{aligned} y(x) &= \int_x^b \frac{y_2(x')y_1(x)}{y_1(x')\partial_x y_2(x') - y_2(x')\partial_x y_1(x')} f(x')dx' + \int_a^x \frac{y_1(x')y_2(x)}{y_1(x')\partial_x y_2(x') - y_2(x')\partial_x y_1(x')} f(x')dx' \\ &= \frac{1}{k \sin(k[b - a])} \left(\int_x^b \sin(k[x' - b]) \sin(k[x - a]) f(x')dx' + \int_a^x \sin(k[x' - a]) \sin(k[x - b]) f(x')dx' \right) \end{aligned}$$