Homework 5:
(Due: 11/27/07)

**1.** Suppose the universe is dominated by a constant potential energy of a scalar field

$$V(\phi) \approx V_0 \equiv \text{constant.}$$

**a)** Solve the Friedmann equation

$$H^2 = \frac{8\pi}{3M_{pl}^2}\rho$$

for $a(\eta)$ where $\eta$ is the conformal time (i.e. $ds^2 = a^2(\eta)[d\eta^2 - |d\vec{x}|^2]$).

**b)** Solve the Friedmann equation for $a(t)$ where $t$ is the proper time of a comoving observer (i.e. $ds^2 = dt^2 - a^2(t)|d\vec{x}|^2$).

**2.** Show that the mapping between the number of efolds $N$ during inflation and the wave vector leaving the horizon is

$$d\ln k = (1-\epsilon)dN.$$

Hint: See page 1 of Lecture 19 and use Friedmann equations.

**3.** Show that

$$v''(\eta) + [k^2 - \frac{2+p}{\eta^2}]v_k(\eta) = 0$$

has a growing solution and a decaying solution for $k/a$ far outside of the horizon during inflation if $p \ll 1$.

**4.** Caricature of reheating:

**a)** Consider the simplest inflationary model studied explicitly in the course:

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2.$$

The classical equation of motion for $\phi(t)$ is

$$\ddot{\phi} + 3H\dot{\phi} + m^2\phi = 0.$$

If one couples a set of light particles to $\phi$, $\phi$ will decay with a width $\Gamma$ and assuming $\Gamma \ll m$, it will enter the equation of motion (in the limit of constant $\Gamma$) as

$$\ddot{\phi} + (3H+\Gamma)\dot{\phi} + m^2\phi = 0$$

From the stress energy tensor of $\phi$, the energy density of $\phi$ is

$$\rho_\phi = \frac{1}{2}(\dot{\phi}^2 + m^2\phi^2).$$

The time derivative of $\rho$ is then

$$\begin{aligned} \dot{\rho_\phi} &= \dot{\phi}\ddot{\phi} + m^2\phi\dot{\phi} \\ &= -(3H+\Gamma)\dot{\phi}^2. \end{aligned} \tag{1}$$

Now, in the limit that $3H + \Gamma \ll m$, virial theorem gives

$$m^2\langle\phi^2\rangle \approx \langle\dot{\phi}^2\rangle$$

where the brackets represent a time average. Hence, we have

$$\langle\rho_\phi\rangle \approx \langle\dot{\phi}^2\rangle$$

if the system is approximately virial. Therefore, we can approximately replace $\dot{\phi}^2$ in Eq. (1) with $\langle\rho_\phi\rangle \approx \rho_\phi$. We thus find

$$\rho_\phi = -(3H+\Gamma)\rho_\phi.$$

Solve this equation assuming $\Gamma$ is a constant (while $H$ is not since inflation has ended).

**b)** The solution to part a) should show that $\rho_\phi$ behaves approximately as $1/a^3$ until time reaches a particular time scale, at which point $\rho_\phi$ decays exponentially. Find that time scale.

**c)** Assuming energy conservation and that all the $\rho_\phi$ energy gets dumped into $\rho_R$ at the time scale of part b), find the reheating temperature assuming $\rho_R = \frac{\pi^2}{30} g_* T_{RH}^4$ with $g_*$ constant.

**5.** Suppose the inflaton is described by the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 - \frac{\lambda}{4}\varphi^4.$$

**a)** What is the exact classical equation of motion for $\varphi(t)$ (i.e. assuming $\varphi$ is homogeneous) when coupled to gravity with an FRW metric: $ds^2 = dt^2 - a^2(t)|d\vec{x}|^2$.

**b)** What is the approximate map between $k/a_0$ (where $a_0$ is the scale factor today) and $\varphi$ if one assumes that the reheating temperature is approximately $10^8$GeV? [Hint: The relationship between the scale factor today and the scale factor $a_{RH}$ at the reheating time $t_{RH}$ is

$$\frac{a_0}{a_{RH}} \approx \frac{T_{RH}}{T_0}\left(\frac{g_{*S}(T_{RH})}{g_{*S}(T_0)}\right)^{1/3}$$

and take $g_{*S}(T_{RH}) \sim 10^2$. The ratio between the scale factor at the end of inflation $a_{\text{end}}$ and the reheating time is

$$\frac{V_{\text{end}}}{\frac{\pi^2}{30} g_*(T_{RH}) T_{RH}^4} = \left(\frac{a_{\text{end}}}{a_{RH}}\right)^{-3}.$$

During inflation, $a(N) = a_i e^{(N-N_i)} = a_{\text{end}}$. The spectrum wave vector $k/a_0$ is fixed by $k/a(N) = H(N)$. Assume slow-roll.]

**c)** How small (order of magnitude) must $\lambda$ be if

$$\sqrt{P_{\mathcal{R}}(k)}|_{k/a_0 \approx 0.005\ \text{Mpc}^{-1}} \sim 10^{-5}?$$

**d)** What is the predicted tensor to scalar ratio $r = (P_h/P_{\mathcal{R}})|_{k/a_0 \approx 0.005\ \text{Mpc}^{-1}}$ for this theory?

**e)** What is the predicted tensor spectral index for this theory?

**6.** Under parity $\{f_{+-} \to f_{-+}, f_{-+} \to f_{+-}\}$ ($f$ is defined in Lecture 22) since parity flips helicity. How does $f_Q + i f_U$ transform under parity?