

## Effective range parametrization of phase shifts for $p$ - $^3\text{He}$ elastic scattering between 0 and 12 MeV

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We have used a modified effective range expansion to parametrize the phase shifts for  $p$ - $^3\text{He}$  elastic scattering over the energy range of 0 to 12 MeV. Energy-dependent phase shifts were determined by fitting a database consisting of 1085 measurements, including 68 new measurements of the spin correlation coefficients  $A_{yy}$ ,  $A_{xx}$ ,  $A_{zz}$ ,  $A_{zx}$ , and  $A_{xz}$ . The addition of the new second-order polarization observables has made it possible to uniquely determine the low-energy phase shifts. The final fit has a  $\chi^2$  per degree of freedom of 0.966, corresponding to a confidence level of 0.778. The global phase shift uncertainties obtained in the present work represent a significant improvement over the results of previous analyses.

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### I. INTRODUCTION

Since the discovery in the early 1960s of excited states in the  $A = 4$  system, there have been numerous attempts to determine the phase shift parameters for  $p$ - $^3\text{He}$  elastic scattering. The earliest elastic scattering measurements consisted mainly of cross section ( $\sigma$ ) and proton analyzing power ( $A_{y0}$ ) points, and as a result the first phase shift analyses resulted in multiple solution sets that all described the data equally well [1-5].

With the construction of the first generation of polarized  $^3\text{He}$  targets, the database was expanded to include  $^3\text{He}$  analyzing power points ( $A_{0y}$ ), as well as the first measurements of spin correlation coefficients by McSherry and Baker [6]. These additional observables served to restrict, but not eliminate, the number of multiple phase shift solutions. While the current generation of polarized targets offers higher polarizations and better long term operational stability, no new second-order polarization observables have been added to the low-energy  $p$ - $^3\text{He}$  database in the last 20 years, and, as a result, the more recent phase shift analyses have continued to find large uncertainties in the fit parameters [7-12].

The purpose of this paper is to present the results of a new phase shift analysis that has led to a significant improvement in the accuracy of the low-energy phase

shift parameters. The new analysis differs from previous work in two important respects. First of all, we have included for the first time the new analyzing power and spin-correlation measurements reported in the preceding paper [13]. We find that the availability of high-quality spin-correlation measurements is essential in eliminating multiple solutions and reducing the overall uncertainties. In addition, for the first time, we have performed an energy-dependent phase shift analysis in which the phase shift parameters are expressed as a function of energy by employing modified effective-range expansions. This procedure results in smoothly varying phase shifts and helps to further reduce the parameter uncertainties.

### II. FORMALISM

#### A. Calculation of the scattering observables

##### 1. $M$ -matrix amplitudes

In order to demonstrate the connection between the phase shift parameters and the scattering observables we adopt a formalism based on the development of La France and Winternitz [14]. In this formalism the scattering matrix  $M$  is written as the sum of six independent terms:

$$M(\mathbf{k}_f, \mathbf{k}_i) = \frac{1}{2} \{ (a+b) + (a-b)(\sigma_1 \cdot \hat{n})(\sigma_2 \cdot \hat{n}) + (c+d)(\sigma_1 \cdot \hat{m})(\sigma_2 \cdot \hat{m}) + (c-d)(\sigma_1 \cdot \hat{l})(\sigma_2 \cdot \hat{l}) + e(\sigma_1 + \sigma_2) \cdot \hat{n} + f(\sigma_1 - \sigma_2) \cdot \hat{n} \}, \quad (1)$$

where the unit vectors  $\hat{l}$ ,  $\hat{m}$ , and  $\hat{n}$  are defined according to

$$\hat{l} = \frac{\mathbf{k}_f + \mathbf{k}_i}{|\mathbf{k}_f + \mathbf{k}_i|}, \quad \hat{m} = \frac{\mathbf{k}_f - \mathbf{k}_i}{|\mathbf{k}_f - \mathbf{k}_i|}, \quad \hat{n} = \frac{\mathbf{k}_f \times \mathbf{k}_i}{|\mathbf{k}_f \times \mathbf{k}_i|}. \quad (2)$$

In Eq. (1),  $\sigma_1$  and  $\sigma_2$  are Pauli operators that act on the

spin wave functions of the proton and the  $^3\text{He}$  nucleus, respectively. The six complex amplitudes  $a$ ,  $b$ ,  $c$ ,  $d$ ,  $e$ , and  $f$  are functions of the reaction energy ( $|k|^2 = |\mathbf{k}_i|^2 = |\mathbf{k}_f|^2$ ) and the scattering angle [ $\theta = \cos^{-1}(\hat{\mathbf{k}}_i \cdot \hat{\mathbf{k}}_f)$ ].

The connection between the  $M$  matrix and the observables (see Ref. [13]) is given by [14],

$$\sigma_0 A_{ik} = \frac{1}{4} \text{Tr} [M \sigma_{1i} \sigma_{2k} M^\dagger], \quad (3)$$

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where  $\sigma_0$  is the unpolarized differential cross section,

$$\sigma_0 = \frac{1}{4} \text{Tr} [MM^\dagger]. \quad (4)$$

Note that in Eq. (3), the analyzing powers  $A_{0y}$  and  $A_{y0}$  are obtained by replacing the appropriate Pauli operator by the unit matrix.

Combining Eqs. (1) and (3), one finds that the elastic scattering observables are given by [14]

$$\sigma_0 = \frac{1}{2} (|a|^2 + |b|^2 + |c|^2 + |d|^2 + |e|^2 + |f|^2), \quad (5a)$$

$$\sigma_0 A_{y0} = \text{Re}(a^*e + b^*f), \quad (5b)$$

$$\sigma_0 A_{0y} = \text{Re}(a^*e - b^*f), \quad (5c)$$

$$\sigma_0 A_{yy} = \frac{1}{2} (|a|^2 - |b|^2 - |c|^2 + |d|^2 + |e|^2 - |f|^2), \quad (5d)$$

$$\sigma_0 A_{xz} = -\text{Re}(a^*d) \sin \theta + \text{Im}(c^*f) - \text{Im}(d^*e) \cos \theta, \quad (5e)$$

$$\sigma_0 A_{zx} = -\text{Re}(a^*d) \sin \theta - \text{Im}(c^*f) - \text{Im}(d^*e) \cos \theta, \quad (5f)$$

$$M_{s'\nu',s\nu} = \frac{i\sqrt{\pi}}{k} \left[ iC(\theta) \delta_{ss'} \delta_{\nu\nu'} + \sum_{jll'} \sqrt{2l+1} (sl\nu 0 | j\nu) \langle s'l'\nu'\nu - \nu' | j\nu \rangle i^{l-l'} U_{l's',ls}^j Y_{l'\nu-\nu'}^{l-l'}(\theta, 0) \right]. \quad (6)$$

The c.m. wave number  $k$  is given by [1]

$$k = \frac{\mu v}{\hbar}, \quad (7)$$

where  $v$  is the relative velocity of the two particles, and  $\mu$  is the reduced mass of the system. The quantity  $C(\theta)$  is the Coulomb scattering amplitude [16],

$$C(\theta) = (4\pi)^{-\frac{1}{2}} \eta \csc^2 \left( \frac{\theta}{2} \right) e^{i\eta \ln \csc^2 \left( \frac{\theta}{2} \right)}, \quad (8)$$

where  $\eta$  is the Coulomb penetration factor [1],

$$\eta = \frac{Z_1 Z_2 e^2}{\hbar v}. \quad (9)$$

The  $Y_l^\nu(\theta, 0)$  are the normalized spherical harmonics of Condon and Shortley [17].

The factor of  $i^{l-l'}$ , which does not appear in the expressions used by Seyler [15] and by Lane and Thomas [16], is included here so that the  $M$  matrix behaves properly under time reversal [1, 18].

The matrix  $U_{l's',ls}^j$  can be expressed in terms of the phase shift parameters connecting the incoming and outgoing partial waves with the same total angular momentum  $j$ . The indices  $l$  and  $l'$  refer, respectively, to the incoming and outgoing orbital angular momentum of the system. Following Tombrello [1], we have

$$U_{l's',ls}^j = e^{i(\alpha_l + \alpha_{l'})} (\delta_{l,l'} \delta_{s,s'} - S_{l's',ls}^j) = U_{ls,l's'}^j, \quad (10)$$

where the  $\alpha$ 's are the modified Coulomb phase shifts,

$$\alpha_0 = 0, \quad \alpha_l = \sum_{s=1}^l \arctan(\eta/s), \quad (11)$$

and where  $S_{l's',ls}^j$  is the scattering matrix.

For the mixed angular momentum states, the  $S$  matrix is parametrized according to the convention of Blatt and Biedenharn [19],

$$\sigma_0 A_{xx} = \text{Re}(a^*d) \cos \theta + \text{Re}(b^*c) - \text{Im}(d^*e) \sin \theta, \quad (5g)$$

$$\sigma_0 A_{zz} = -\text{Re}(a^*d) \cos \theta + \text{Re}(b^*c) + \text{Im}(d^*e) \sin \theta, \quad (5h)$$

where  $\text{Re}$  and  $\text{Im}$  refer to the real and imaginary parts of the complex argument.

## 2. $M$ -matrix elements

In channel-spin representation the elements of the  $M$  matrix are written as  $M_{s'\nu',s\nu}$ , where  $s, \nu$  and  $s', \nu'$  denote the channel-spin quantum numbers for the initial and final states, respectively. For the channel spin we employ the coupling order  $\mathbf{s} = \mathbf{s}_1 + \mathbf{s}_2$ , where as before  $\mathbf{s}_1$  is the proton spin and  $\mathbf{s}_2$  is the  ${}^3\text{He}$  spin. A given matrix element can be expanded in partial waves according to [15]

$$S = \begin{pmatrix} \cos^2 \epsilon e^{2i\delta_1} + \sin^2 \epsilon e^{2i\delta_2} & \frac{1}{2} \sin 2\epsilon [e^{2i\delta_1} - e^{2i\delta_2}] \\ \frac{1}{2} \sin 2\epsilon [e^{2i\delta_1} - e^{2i\delta_2}] & \cos^2 \epsilon e^{2i\delta_2} + \sin^2 \epsilon e^{2i\delta_1} \end{pmatrix}. \quad (12)$$

Here  $\delta_1$  and  $\delta_2$  are the "eigen phase shifts" and  $\epsilon \equiv \epsilon(j\pi)$  is the mixing parameter connecting the two states with the same total angular momentum  $j$  and parity  $\pi$ . In order that the sign of the mixing parameter be well defined, we adopt the convention that the phase shift  $\delta_1$  refers to the parameter with the lesser value of either  $l$  or  $s$ .

Finally, to make the connection between the phase shifts and the scattering observables complete, we give the relationship between the  $M$ -matrix amplitudes and the  $M$ -matrix elements [14]:

$$a = \frac{1}{2} (M_{1010} + M_{1111} - M_{1-111}), \quad (13a)$$

$$b = \frac{1}{2} (M_{0000} + M_{1111} + M_{1-111}), \quad (13b)$$

$$c = \frac{1}{2} (-M_{0000} + M_{1111} + M_{1-111}), \quad (13c)$$

$$d = -\frac{1}{\sqrt{2} \sin \theta} (M_{1110} + M_{1011}), \quad (13d)$$

$$e = \frac{i}{\sqrt{2}} (M_{1110} - M_{1011}), \quad (13e)$$

$$f = -i\sqrt{2} M_{1100}. \quad (13f)$$

## B. Effective range expansion

In order to perform a global fit of the measurements obtained over a range of energies, we need to express the phase shift parameters as functions of the scattering energy  $E$ . This is done by using a modified effective range expansion [20] in which each phase shift,  $\delta_{ls}^j$ , is parametrized through a power series in  $k^2$ :

$$C_l^2(\eta)k^{2l+1} \left[ \cot \delta_{ls}^j + 2\eta H(\eta)/C_0^2(\eta) \right] = \sum_{n=0}^{\infty} a_n^{js} k^{2n}, \quad (14)$$

with the following definitions:

$$C_0^2(\eta) = \frac{2\pi\eta}{e^{2\pi\eta} - 1}, \quad (15)$$

$$C_l^2(\eta) = C_{l-1}^2(\eta) \left( 1 + \frac{\eta^2}{l^2} \right), \quad (16)$$

$$H(\eta) = \eta^2 \sum_{s=1}^{\infty} \frac{1}{s(s^2 + \eta^2)} - \ln \eta - \gamma, \quad (17)$$

$$\gamma = 0.577216 \text{ (Euler's constant)}. \quad (18)$$

Strictly speaking, Eq. (14) is correct only for the case in which there is no coupling between angular momentum states. However, since the formula produces an energy dependence that reflects the presence of the Coulomb and angular momentum barriers, it represents a reasonable choice for a phenomenological parametrization of the phase shifts.

Choosing a reasonable form for the expansion of the mixing parameters is more difficult, since their behavior is not as well understood as that of the phase shifts. We note, first of all, that the value of the mixing parameter is inherently ambiguous by  $\pm 90^\circ$ , since in Eq. (12), the operation

$$\delta'_1 = \delta_2, \quad \delta'_2 = \delta_1, \quad \text{and} \quad \epsilon' = \epsilon \pm 90^\circ, \quad (19)$$

leaves the  $S$  matrix unchanged. This implies that as  $E$  goes to zero, the value of  $\epsilon$  can approach either  $0^\circ$  or  $90^\circ$ . Presumably, the sensible choice is to pick the solution in which  $\epsilon$  goes to zero at low energies. Blatt and Biedenharn [21, 22] have shown that, for  $n$ - $p$  elastic scattering in the limit of low energies, the  ${}^3S_1$ - ${}^3D_1$  mixing parameter  $\epsilon(1^+)$  varies with the wave number according to

$$\tan \epsilon \cong ak^2, \quad (20)$$

where  $a$  is a constant. It is presently unclear how this behavior is modified by the presence of the Coulomb interaction, or whether the same result applies for the other mixing parameters. We have chosen to adopt the following simple power series expansion for all of the mixing parameters:

$$\tan \epsilon(j^\pi) = \sum_{i=1}^{\infty} a_i^j k^{2i} \quad (21)$$

While this formula obviously does not include Coulomb effects, we find that it provides a convenient way of de-

scribing the behavior of the mixing parameters over the 0 to 12 MeV energy range.

### III. THE DATABASE

For the phase shift fits we have used a database that includes most of the available  ${}^3\text{He}(p, p){}^3\text{He}$  measurements. The database, which consists of a total of 1085 measurements, is summarized in Table I. To facilitate the treatment of normalization uncertainties, the data have been divided into 14 groups consisting of measurements that are thought to have common normalization errors. The measurements of groups 1 and 2 are from the experiment described in the preceding paper [13]. The data in group 1 are measurements that depend on the  ${}^3\text{He}$  target polarization ( $A_{0y}$ ,  $A_{xx}$ ,  $A_{yy}$ ,  $A_{zz}$ ,  $A_{xz}$ , and  $A_{zx}$ ), while the measurements of group 2 consist entirely of proton analyzing power points ( $A_{y0}$ ).

Of the data from other authors, several points have been discarded for various reasons. In previous work, Clegg *et al.* [2] encountered some difficulties with the cross section measurements of both Brolley *et al.* [28] and of Lovberg [29]. As a result, the data of Brolley *et al.* obtained from recoil  ${}^3\text{He}$  measurements were not included in the group 12 data set. The forward angle data of Lovberg [29] apparently suffer from problems due to large double scattering contributions and from the inability to resolve protons scattered from contaminant nuclei, and consequently the measurements at  $30^\circ$ ,  $35^\circ$ , and  $40^\circ$  (c.m.) have been omitted from the group 13 data set. Finally, the proton analyzing power measurements of Drigo *et al.* [31] were not included in our database, since these measurements are thought to be faulty based on the  $R$ -matrix analysis of Detomo *et al.* [25].

Several additional data points were rejected on the basis of our own fitting results. After each effective range fit had been completed, the  $\chi^2$  contribution of each individual data point was inspected. If a given measurement was found to always have a large contributing error, this was taken as an indication of a possible systematic error not reflected in the quoted uncertainty. For the data of group 12, it was noted over the course of several fits that the three most forward angle cross section points at 6.5 MeV and the most forward point at 8.3 MeV all had  $\chi^2$  contributions in excess of 20. Similarly, the most forward angle  $A_{y0}$  measurement of group 6 at four energies (1.75, 2.00, 2.25, and 2.75 MeV) all had individual contributions to  $\chi^2$  of greater than 10, indicating errors of greater than  $3\sigma$ . As a result, these eight points were omitted from the database, and are not included in Table I.

After completing the phase shift fits (described below), the effect of these eight data points on the final phase shift values was investigated, and it was found that these points do not have a large effect on the final results. In particular, we find that when these points are included in the database, the phase shift parameters change by only a small fraction (typically 20%) of the final quoted parameter uncertainty.

TABLE I. Data used in the global effective range phase shift fits for the  ${}^3\text{He}(p,p){}^3\text{He}$  reaction. The data set includes a total of 1085 measurements.

Group	Observable	Energies	Number of data	Reference
1	$A_{0y}$	4.01, 5.54, 7.03, 8.52, 10.01	48	[13]
	$A_{yy}$	4.01, 5.54, 7.03, 8.52, 10.01	48	
	$A_{xz}$	5.54	5	
	$A_{zx}$	5.54	5	
	$A_{xx}$	5.54	5	
	$A_{zz}$	5.54	5	
2	$A_{y0}$	4.01, 5.54, 7.03, 8.52, 10.01	183	[13]
3	$A_{y0}$	6.80, 8.82	8	[23]
	$A_{0y}$	6.80, 8.82	8	
	$A_{xx}$	6.80, 8.82	8	
	$A_{yy}$	6.80, 8.82	8	
4	$A_{0y}$	3.86, 4.38, 4.89, 5.90, 6.91, 7.92, 8.93, 9.93, 10.94	34	[7]
	$A_{zz}$	8.80	2	
5	$\sigma$	1.01, 1.60, 2.25, 3.5	36	[24]
6	$A_{y0}$	1.75, 2.00, 2.25, 2.50, 2.75, 2.95, 3.13, 3.36, 3.55, 3.75, 4.00, 4.50	153	[25]
	$A_{0y}$	2.30, 3.00, 4.47, 6.80, 8.80	63	
8	$A_{0y}$	10.5	26	[9]
9	$\sigma$	4.00, 5.51, 6.82, 8.82, 10.77	80	[27]
	$A_{y0}$	4.05, 5.52, 6.83, 8.83, 10.74, 12.79	28	
10	$\sigma$	4.55, 5.51, 6.52, 7.51, 8.51, 9.51, 10.38, 11.48	167	[2]
	$A_{y0}$	4.00, 5.51, 6.82, 8.82, 10.77	69	
12	$\sigma$	6.5, 8.3	58	[28]
13	$\sigma$	9.75	22	[29]
14	$A_{0y}$	12.4	16	[30]

#### IV. FITTING PARAMETERS

To determine which phase shift parameters must be included in order to reproduce the  ${}^3\text{He}(p,p){}^3\text{He}$  database, we observe first of all that at low energies elastic scattering is the dominant reaction channel. The threshold for the breakup reaction  $p + {}^3\text{He} \rightarrow d + 2p$  occurs at a proton energy of 7.3 MeV, while the next reaction channel ( $p + {}^3\text{He} \rightarrow n + 3p$ ) does not open until  $E_p = 10.3$  MeV. Both reactions have very low cross sections at low energies (e.g.,  $\sigma = 5 \mu\text{b}$  for the four-body breakup reaction at 13 MeV [3, 32]), and so it is conventional to ignore the inelastic scattering processes in the energy regime below 20 MeV. Therefore all of the phase shifts are assumed to be real.

For the range of 0 to 12 MeV, all of the  $S$  ( ${}^1S_0$ ,  ${}^3S_1$ ) and  $P$  ( ${}^1P_1$ ,  ${}^3P_2$ ,  ${}^3P_1$ ,  ${}^3P_0$ ) phase shifts are clearly necessary for an accurate reproduction of the data. It is less clear how many of the higher partial waves are needed for energies up to 12 MeV. We have found that the  ${}^3D_j$  phase shifts are quite small in absolute magnitude and vary little between the different  $j$  values. Consequently, in our fits we take the three triplet  $D$  waves to be equal, so that only a single adjustable parameter,  ${}^3D_j$ , is needed. Similarly, we include a single  ${}^3F_j$  parameter to represent all of the triplet  $F$  waves. While this phase shift was always small

(maximum values were on the order of  $3^\circ$ ), it was indispensable for obtaining a good fit to the data. For the corresponding singlet waves, it was found that the  ${}^1D_2$  was important for a successful fit. On the other hand, the  ${}^1F_3$  phase shift never noticeably improved the fit results, and therefore this partial wave was not included in the final search.

As for the mixing parameters, it has long been known that the  $\epsilon(1^-)$  plays an important role. The importance of the remaining mixing parameters is less clear, and different authors have included different parameters in their work. Szaloky and Seiler [10] have used  $\epsilon(1^+)$  and  $\epsilon(2^+)$  in addition to the  $\epsilon(1^-)$ , while Heiss and Hackenbroich [8] have found that only  $\epsilon(2^-)$  was appreciable at higher energies. Our best results (see Sec. V A) were obtained by including the  $\epsilon(1^-)$ ,  $\epsilon(1^+)$ , and  $\epsilon(2^-)$  mixing parameters.

The assumed energy dependence of the various phase shifts and mixing parameters has already been described in Eqs. (14) and (21). For each phase shift and mixing parameter the power series was truncated to include only three expansion coefficients. With nine phase shifts and three mixing parameters, this brings the total number of fitting parameters to 36.

The overall normalization of each separate data set was handled as follows. In each data group  $k$  (see Table I) we have a set of measurements  $y_i^k$  with errors  $dy_i^k$ , where

the index  $i$  refers to the individual measurements. If we let  $\nu_k$  represent the scale factor for the measurements of group  $k$ , then the group  $\chi^2$  may be written as [33]

$$\chi_k^2 = \min_{\nu_k} \sum_i \left( \frac{\nu_k f_i^k - y_i^k}{dy_i^k} \right)^2 + \left( \frac{\nu_k - \nu_0}{d\nu_k} \right)^2, \quad (22)$$

where the  $f_i^k$  are the predictions from the phase shift fit. The last term in Eq. (22) represents the contribution to the overall  $\chi^2$  from the renormalization of the data. Here  $\nu_0$ , which always has the value 1, represents the experimenter's estimate of the required renormalization, while  $d\nu_k$  (typically 0.05) is the normalization uncertainty.

The procedure we follow is to first vary the phase shift parameters to minimize the overall  $\chi^2$ :

$$\chi^2 = \sum_{k=1}^{14} \chi_k^2, \quad (23)$$

with the group scale factors fixed at  $\nu_k = 1$ . Once the phase shifts have been determined the scale factors are adjusted to further minimize  $\chi_k^2$ . This last step is easily accomplished analytically, with the result

$$\nu_k = \left[ \sum_i \frac{y_i^k f_i^k}{(dy_i^k)^2} + \frac{\nu_0}{d\nu_k^2} \right] / \left[ \sum_i \left( \frac{f_i^k}{dy_i^k} \right)^2 + \frac{1}{d\nu_k^2} \right]. \quad (24)$$

It should be emphasized that the group normalizations are not fitting parameters but rather scale factors calculated after the effective range coefficients have been determined.

## V. PHASE SHIFT FITS

### A. General comments

With the addition of the spin correlation measurements from Ref. [13], it was immediately clear that the

new data could not be adequately described with the  $\epsilon(1^-)$  mixing parameter alone. In particular, over the energy range of 4 to 10 MeV the phase shift predictions of  $A_{yy}$  were generally 50 to 100% smaller than the measurements in the region from  $60^\circ$  to  $120^\circ$  in the c.m. It was also clear that the back-angle  $A_{yy}$  measurements play an important role in the phase shift determination; small variations between potential solution sets produced large differences (i.e., factors of 2 or more) in the predicted values of  $A_{yy}$  in the angular range of  $140^\circ$  to  $180^\circ$ .

Effective range searches were ultimately attempted using combinations of the mixing parameters  $\epsilon(1^-)$ ,  $\epsilon(1^+)$ ,  $\epsilon(2^-)$ , and  $\epsilon(2^+)$ . The parameter set containing the  $\epsilon(1^-)$ ,  $\epsilon(1^+)$ , and  $\epsilon(2^-)$  eventually produced a total  $\chi^2$  that was over 9% lower than the parameter set that included the  $\epsilon(1^-)$ ,  $\epsilon(1^+)$ , and  $\epsilon(2^+)$  mixing parameters. For this reason the  $\epsilon(2^+)$  was not retained in the final fit. Both the  $\epsilon(1^+)$  and  $\epsilon(2^-)$  were considered to be indispensable for a reasonable fit to the data, even though their magnitudes were never greater than  $5^\circ$ .

### B. Results

The final phase shift solutions are shown in Figs. 1-3. For comparison, our results are plotted along with the solutions of McSherry and Baker [7], Szaloky and Seiler [10], and Beltramin *et al.* [12]. In Fig. 1 we see that the agreement is quite good for the  $^3S_1$  phase shift. The  $^1S_0$  phase shift appears to be less well determined, as can be seen from the wide variations of the solutions shown.

The results for the  $P$ -wave phase shifts are shown in Fig. 2. In comparing these parameters one needs to insure that all results correspond to the same branch of Eq. (19). In Refs. [7] and [10], the reported  $\epsilon(1^-)$  mixing parameters are close to  $90^\circ$ . Consequently, in preparing the graphs we have shifted the  $\epsilon(1^-)$  values by  $90^\circ$  and interchanged the  $^3P_1$  and  $^1P_1$  phase shifts reported in those references. Our results for the  $^3P$  phase shifts agree quite well with the previous work. However, for the

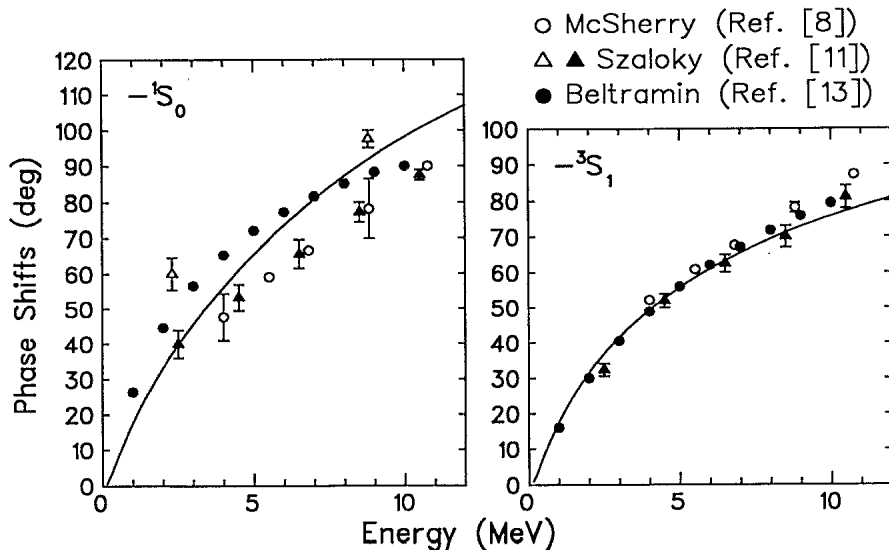


FIG. 1. The effective range  $S$ -wave phase shift solutions (solid curves), shown along with the results of McSherry and Baker [7], Szaloky and Seiler [10], and Beltramin *et al.* [12]. The open triangles are taken from Table 3 of Ref. [10], while the solid triangles correspond to the error bands displayed in Fig. 1 of Ref. [10]. In this latter case the data point shows the midpoint of the error band, while the error bar corresponds to the width of the band. Points that are shown without error bars correspond to results published with no quoted uncertainty.

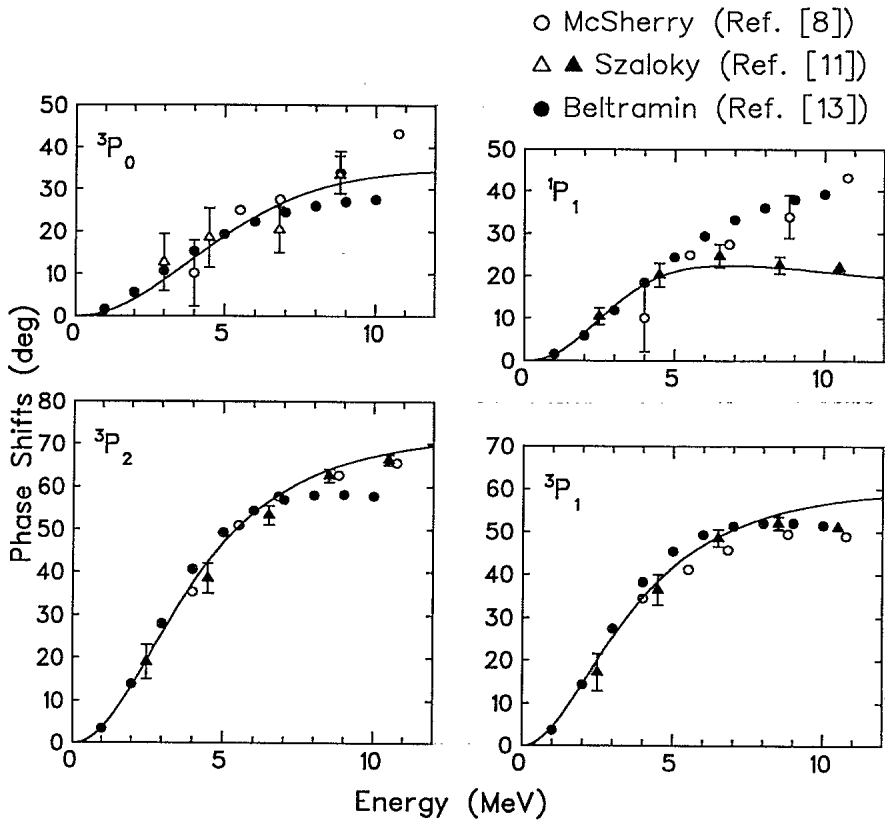


FIG. 2. The effective range  $P$ -wave phase shift solutions (solid curves), shown along with the results of McSherry and Baker [7], Szaloky and Seiler [10], and Beltramin *et al.* [12]. See the caption of Fig. 1 for details.

$^1P_1$  there are substantial differences for energies greater than 5 MeV. The remaining phase shifts ( $^1D_2$ ,  $^3D_3$ , and  $^3F_3$ ) and mixing parameters [ $\epsilon(1^-)$ ,  $\epsilon(1^+)$ , and  $\epsilon(2^-)$ ] are shown in Fig. 3.

The final  $\chi^2$  of the fit was 1013.77, which for 1085 points and 36 parameters results in a  $\chi^2$  per degree of

freedom of 0.966. This in turn yields a confidence level of 0.778. The best fit group normalizations  $\nu_k$  ranged from 0.957 for group 1 to 1.076 for group 4. It should be noted here that the excellent  $\chi^2$  values result in part from the fact (see Sec. III) that we discarded several points which had anomalously high  $\chi^2$  contributions.

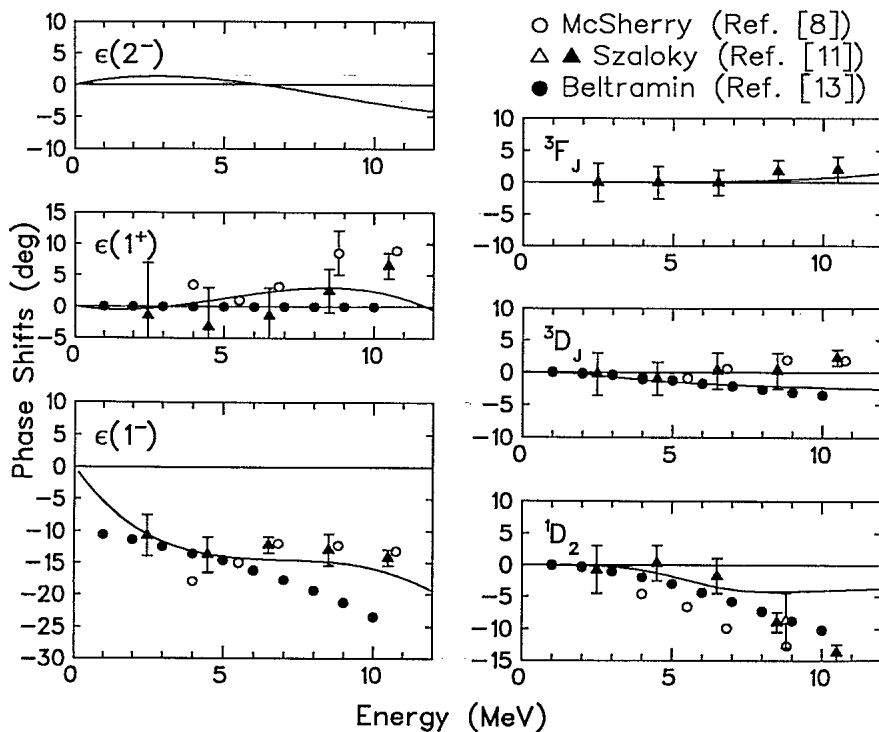


FIG. 3. The effective range solutions for the  $D$ - and  $F$ -wave phase shifts and for the three mixing parameters. See the caption of Fig. 1 for details. Our results for the  $^3D_3$  phase shift are compared with the  $^3D_2$  phase shift results from Refs. [7, 12].

### C. Phase shift uncertainties

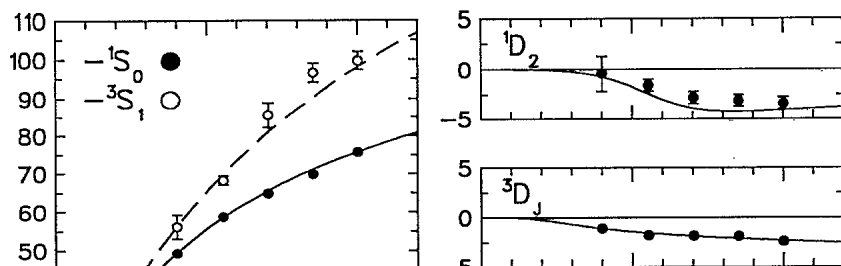
Having established a new phase shift set for the energy range 0 to 12 MeV, it is desirable to determine how well the phase shift parameters have been determined. In principle, one could try to derive error bands for the global phase shift solutions using the effective range parametrizations of Eqs. (14) and (21). However, we have chosen not to use this approach for two reasons. First of all, in spite of the fact that the effective range formulas have a basis in theory, they do not necessarily give an accurate representation of the true phase shift energy dependence. As a result, the true phase shift errors may well be larger than the range of values allowed within the context of this particular parametrization. Second, the usual approach of deriving parameter uncertainties from the error matrix takes into account only the statistical errors in the measurements. For situations in which the database contains a very large number of measurements, it may well be the case that systematic errors in the data set play a more important role than the statistical errors. In view of the complexity involved in deriving meaningful energy-dependent error bands, we adopt a more modest goal, namely, to obtain a single overall error estimate for each phase shift parameter that is meant to be representative of the uncertainty over the entire energy range.

The approach we use to estimate the parameter un-

certainties involves performing a series of single-energy phase shift fits. The single-energy fits were carried out at the five energies (4.01, 5.54, 7.03, 8.52, and 10.01 MeV) of the experiment described in the preceding paper [13]. For each of the single-energy fits, the starting phase shifts were taken from the effective range solution shown in Figs. 1-3. The nine phase shifts and the three mixing parameters were then treated as free parameters which were adjusted to minimize chi-square. At the conclusion of each fit, the statistical errors in the individual parameters were determined in the usual way by extracting the diagonal elements of the error matrix. The results of the single-energy phase shift fits and the corresponding  $1\sigma$  statistical errors are shown in Fig. 4 along with the effective range results.

As can be seen from Fig. 4, in most cases the agreement between the single-energy fits and the effective range results is quite good. There are some exceptions, namely for the  $^1S_0$ ,  $^1P_1$ ,  $\epsilon(1^+)$ , and  $\epsilon(2^-)$  parameters, where at the energies of 7.03 and 8.52 MeV, the single-energy results differ from the global fit by more than one standard deviation.

We believe that these relatively large differences probably reflect the presence of systematic errors in the measurements. To illustrate the point we show in Fig. 5 the predicted observables obtained from the global fit and from the single-energy fit at 8.52 MeV. To the eye the



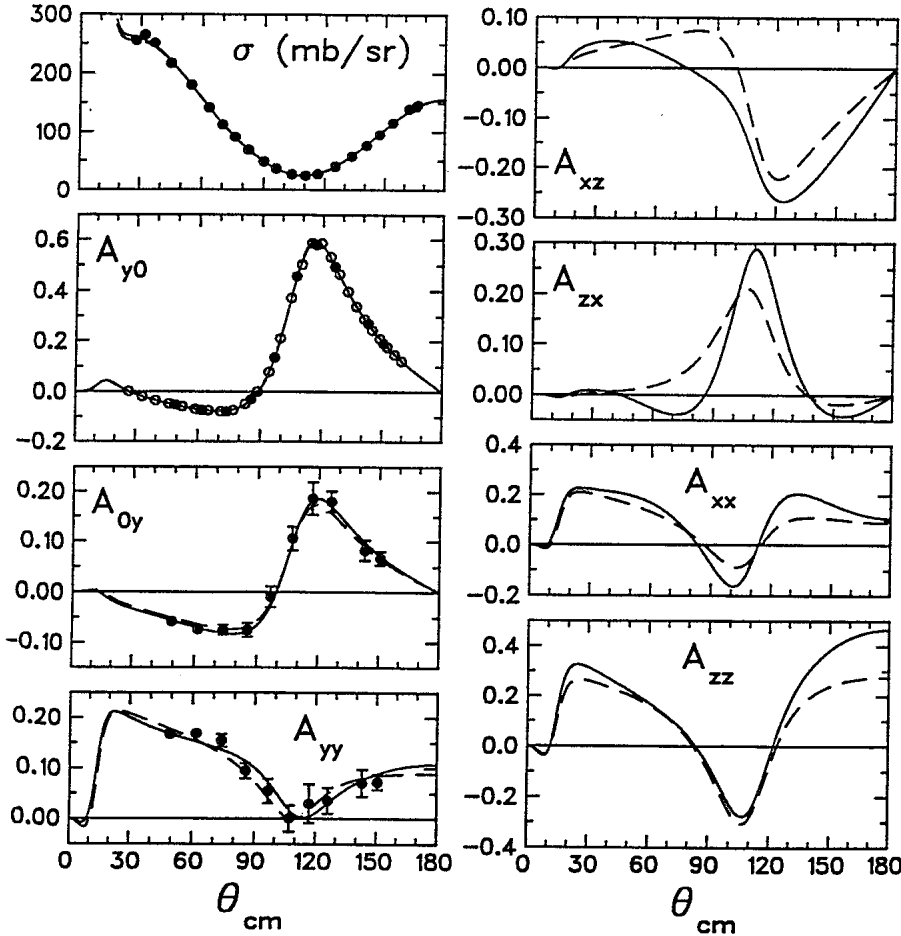


FIG. 5. Predictions of the cross section, analyzing powers, and spin correlation coefficients at 8.52 MeV obtained from the effective range solutions (dashed lines) and from the single-energy fits (solid lines).

two fits appear to be quite comparable in quality. It turns out that the main difference between the two fits (as far as  $\chi^2$  is concerned) occurs in the proton analyzing power  $A_{y0}$  in the angle range between  $80^\circ$  and  $110^\circ$  where the two curves differ systematically by about one standard deviation. Since there are many data points in the region, one can easily imagine that a small systematic error in the angle determination (on the order of  $0.1^\circ$ ) would be capable of skewing the single-energy phase shift results. Of course, the global fit should be affected as well, but since the database is much larger and the fit is constrained by the effective range formulas the effect should be much smaller in this case.

Figure 5 also illustrates the value of spin correlation measurements for determining the correct phase shift solution. In particular, we see that even relatively crude measurements of  $A_{xz}$  and  $A_{zx}$  would go a long way to resolve the differences between the two fits and thereby minimize the phase shift uncertainties.

We shall now describe how we use the results of the single-energy fits to estimate the overall uncertainty in the global phase shifts. The effective range and single-energy phase shift solutions will be represented by  $\delta^{\text{eff}}$  and  $\delta^{\text{se}}$ , respectively, and the statistical uncertainty derived from the single-energy error matrix will be represented by  $d\delta^{\text{se}}$ . For purposes of estimating the systematic errors, the phase shift parameters may be divided into two groups. For most of the parameters the difference between the  $\delta^{\text{eff}}$  and  $\delta^{\text{se}}$  is comparable in magnitude to

the statistical uncertainty. In other words, the average deviation,

$$\sigma = \frac{1}{5} \sum_{j=1}^5 \left| \frac{\delta^{\text{eff}}(E_j) - \delta^{\text{se}}(E_j)}{d\delta^{\text{se}}(E_j)} \right| \quad (25)$$

is close to 1. We take this to mean that, for these parameters, the uncertainties are dominated by statistical errors, and therefore we take the uncertainty to be just the mean single-energy statistical error

TABLE II. Estimates of the overall phase shift uncertainties.

Parameter	Overall error
$^1S_0$	$3.5^\circ$
$^3S_1$	$0.6^\circ$
$^1P_1$	$1.5^\circ$
$^3P_2$	$0.5^\circ$
$^3P_1$	$0.5^\circ$
$^3P_0$	$0.6^\circ$
$^1D_2$	$1.0^\circ$
$^3D_2$	$0.2^\circ$
$^3F_2$	$0.1^\circ$
$\epsilon(1^-)$	$0.6^\circ$
$\epsilon(1^+)$	$1.5^\circ$
$\epsilon(2^-)$	$2.4^\circ$



$$d\bar{\delta}^{se} = \frac{1}{5} \sum_{j=1}^5 d\delta^{se}(E_j). \quad (26)$$

For the remaining parameters [ $^1S_0$ ,  $^1P_1$ ,  $\epsilon(1^+)$ , and  $\epsilon(2^-)$ ]  $\sigma$  is close to 2 which suggests that systematic errors are important. For these parameters the uncertainty is taken to be  $2d\bar{\delta}^{se}$ . The final parameter error estimates are listed in Table II.

## VI. CONCLUSION

We have obtained a new set of energy dependent phase shifts for  $p$ - $^3\text{He}$  elastic scattering in the energy range 0 to 12 MeV. The phase shifts and mixing parameters were parametrized using modified effective range formulas, and this has proven to be a useful approach for modeling the phase shifts over the low-energy region. We have obtained an excellent fit (reduced  $\chi^2$  of 0.966) to

the global database which contains a total of 1085 measurements.

The new phase shifts are considerably more accurate than previous determinations. The use of an energy dependent parametrization (as opposed to single energy fits) and the addition of new spin correlation measurements have been instrumental in eliminating ambiguities and reducing the phase shift uncertainties. Extending the current analysis to higher energies would presumably be straightforward with the addition of spin correlation measurements in this energy region.

## ACKNOWLEDGMENTS

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