Watson's theorem for low-energy *p-d* radiative capture

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The use of Watson's theorem in the analysis of *p-d* radiative capture measurements at low energies is discussed. The principle of Watson's theorem is outlined, and a detailed description of how the theorem can be used in a matrix element analysis of radiative capture data is presented. It is shown that with Watson's theorem it is possible to reduce the number of unknown parameters in a matrix element analysis by essentially a factor of 2. This is done by employing a representation in which the capture matrix elements are all real. The phase information needed to construct the reaction amplitudes is then obtained from a separate phase shift analysis of elastic scattering data. Details concerning the extension of Watson's theorem to situations in which there is mixing between angular momentum states are given. The paper presents a consistent formulation which facilitates the simultaneous analysis of the elastic scattering and radiative capture channels. [S0556-2813(99)01304-7]

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I. INTRODUCTION

The radiative capture reaction $p+d\rightarrow{}^3\mathrm{He}+\gamma$ is of interest for a number of reasons. It is known, for example, that the deuteron tensor analyzing powers are sensitive to *D*-state components in the ^{3}He wave function [1], and measurements of these quantities have often been used in the past to extract *D*-state information [2–4]. There has also been a substantial amount of interest in obtaining information about the $M1$ contributions to the reaction process (see for example Refs. [5,6]). This interest arises in part from the desire to understand the role of meson exchange processes. Finally, the *p-d* capture reaction is of special importance since it is one of the basic processes that can be studied in the three-nucleon system. Recent theoretical and computational advances have made it possible to carry calculations in the $A=3$ system which are quantum mechanically exact, and as a result, *p-d* capture measurements have taken on added importance since they can now be used to test, at a rather fundamental level, our understanding of the nuclear three-body problem.

In radiative capture reactions, one can often learn much from measurements of cross sections or analyzing powers at only a few angles. However, it is well understood that information of a more fundamental nature can be obtained if one has a data set sufficiently complete to allow the extraction of individual reaction matrix elements. For capture reactions that have a simple spin-parity structure, the determination of

more complicated when the spin structure is such that mixing between angular momentum states is allowed in the elastic scattering channel. Nevertheless, as we shall demonstrate, the basic principle of the Watson theorem still holds, provided that one defines the matrix elements appropriately.

The application of Watson's theorem to radiative capture is essentially equivalent to carrying out a simultaneous partial-wave analysis of the elastic scattering and reaction channels. As such, it is clear that one must be exceedingly careful about phase conventions, coupling schemes for angular momenta, and so on. For this reason, the discussion that follows is quite detailed, with clear and rigorous definitions presented for all relevant quantities.

We begin in Sec. II by defining the transition matrix for the radiative capture reaction, introducing the multipole expansion of the interaction Hamiltonian, and defining the phase conventions for the angular momentum states. In Sec. III we introduce the elastic scattering wave function and demonstrate Watson's theorem for the simple case of no coupling between angular momentum states. The generalization of Watson's theorem to situations in which angular momentum mixing is allowed is presented in Sec. IV. Section V presents a discussion of a few subtle points, and the final conclusions are given in Sec. VI.

II. THE RADIATIVE CAPTURE REACTION

A. The starting point

The radiative capture reaction is described as a process in which transitions between the initial state χ, and the final state μ, are induced by the electromagnetic interaction, $H_{\rm int}$. The initial and final states are eigenstates of a Hamiltonian H_0, and the full Hamiltonian of the system is

$$H = H_0 + H_{\rm int}. \tag{2.1}$$

The interaction Hamiltonian is given by the usual expression

$$H_{\rm int} = -\frac{1}{c}\int \mathbf{J}(\mathbf{r})\cdot\mathbf{A}(\mathbf{r})d^3r, \tag{2.2}$$

where $\mathbf{J}(\mathbf{r})$ is the current density and $\mathbf{A}(\mathbf{r})$ is the vector potential.

In a nuclear system, $H_{\rm int}$ takes the form [10]

$$H_{\rm int} = -\frac{e\hbar}{2mc}\sum_n \left[2g_{ln}\mathbf{p_n}\cdot\mathbf{A}(\mathbf{r_n}) + g_{sn}\mathbf{s_n}\cdot\mathbf{H}(\mathbf{r_n})\right], \tag{2.3}$$

where

$$\mathbf{H}(\mathbf{r}) \equiv \boldsymbol{\nabla}\times\mathbf{A}(\mathbf{r}). \tag{2.4}$$

In Eq. (2.3) the sum is over all nucleons, and the operators $\mathbf{p}$ and $\mathbf{s}$ have the definitions

$$\mathbf{p} = -i\boldsymbol{\nabla} \tag{2.5}$$

and

$$\mathbf{s} = \frac{1}{2}\boldsymbol{\sigma}, \tag{2.6}$$

where the components of $\boldsymbol{\sigma}$ are the Pauli matrices. The quantity g_{ln} takes on the values 0 and 1 for neutrons and protons, respectively, while the corresponding values of g_{sn} are -3.826 and 5.585.

Since the coupling between the initial and final states of the system is weak, the transition rates may be calculated in time-dependent perturbation theory. As seen in the following section, this allows us to find the connection between the matrix elements of $H_{\rm int}$ and the radiative capture observables.

B. The transition matrix

Our goal is to find the transition matrix T_{fi} for the radiative capture process. Given T_{fi}

C. Reduction of H_{int}

The first step in obtaining the multipole expansion of the matrix element in Eq. (2.11) is to simplify H_{int}. Strictly speaking, the quantities $\mathbf{A}(\mathbf{r})$ and $\mathbf{H}(\mathbf{r})$ in Eq. (2.3) should be treated as field operators. However, for present purposes it is preferable to adopt the semiclassical approach and refer the reader to Ref. [10] for a discussion of the more subtle aspects of the full quantum treatment.

In the semiclassical approach the electromagnetic wave is described by a vector potential of the form

$$\begin{aligned}\mathbf{A}(\mathbf{r},t) &= A_0 \mathrm{Re}[\hat{\boldsymbol{\epsilon}} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}] \\ &= \frac{A_0}{2}[\hat{\boldsymbol{\epsilon}} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} + \hat{\boldsymbol{\epsilon}}^* e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}]. \qquad (2.12)\end{aligned}$$

Following Ref. [10] we may then divide H_{int} into two parts,

$$H_{\text{int}} = \frac{A_0}{2}[H_a($$

$$\langle \phi | B | \psi \rangle = \langle \theta_T \phi | \theta_T B \theta_T^{-1} | \theta_T \psi \rangle^* . \tag{2.24}$$

We now apply this formula to the matrix element of $T_{L\lambda}$ taken between two eigenstates of the total angular momentum. One can easily demonstrate that the electric and the magnetic multipole operators both transform according to the rule

$$\theta_T T_{L\lambda} \theta_T^{-1} = (-)^{L-\lambda} T_{L-\lambda} . \tag{2.25}$$

Let us assume that for the angular momentum states we adopt phase conventions so that the time reversal transformation is

$$\theta_T | j,m \rangle = (-)^{j-m} | j,-m \rangle . \tag{2.26}$$

With our definition of operator θ_T this equation is satisfied by the conventional spin angular momentum eigenstates. For eigenstates of orbital angular momentum Eq. (2.26) will be satisfied if we take

$$| l,m \rangle \equiv i^l Y_l^m , \tag{2.27}$$

where Y_l^m is a spherical harmonic defined with the usual phase convention [11]. Finally we note that if $|j_1,m_1\rangle$ and $|j_2,m_2\rangle$ satisfy Eq. (2.26), then

$$| j,m \rangle = \sum_{m_1,m_2} | j_1,m_1 \rangle | j_2,m_2 \rangle \langle j_1 m_1, j_2 m_2 | j m \rangle \tag{2.28}$$

does also.

tions to the Schrödinger equation are eigenfunctions of L^2 and S^2, so that l and s are good quantum numbers. In addition we assume throughout that there are no open reaction channels. It follows that the most general solution of the Schrödinger equation in a given angular momentum state will consist of ingoing and outgoing spherical waves in the asymptotic region. In other words the solutions at large r will be characterized by radial wave functions that can be written in the form

$$g_l(r) \rightarrow \frac{1}{2ik_i r}[be^{i[k_i r - l(\pi/2)]} - ae^{-i[k_i r - l(\pi/2)]}], \quad (3.8)$$

where a and b are constants. The notation we adopt in Eq. (3.8) and in the following discussion is that of Blatt and Biedenharn [12].

Since the outgoing wave evolves from the ingoing wave, it follows that the outgoing wave amplitude b must be a function of the ingoing wave amplitude a. Then, since the Schrödinger equation is linear, we conclude that a and b must be related by an equation of the form

$$b = Sa, \quad (3.9)$$

where the proportionality constant S is the S-matrix element for the angular momentum state in question. If there are no open reaction channels then flux conservation requires $|a| = |b|$ and we may write

$$S = e^{2i\delta}, \quad (3.10)$$

$$T_{\lambda\sigma,\nu_p\nu_d} = -4\pi\left[\frac{\mu_i\omega}{2\pi\hbar^2 c k_i}\right]^{1/2} \sum_{\substack{J,l,s,L \\ M,\nu,m}} \langle s_p\nu_p, s_d\nu_d | s\nu\rangle$$
$$\times\langle lm, s\nu | JM\rangle e^{i\delta_J^{ls}}$$
$$\times\langle ls; JM | T^e_{L\lambda} + \lambda T^m_{L\lambda} | \phi^\sigma_{\rm He}\rangle^* Y^{m*}_l(\hat{\boldsymbol{k}}_i), \quad (3.18)$$

where $\phi_{\rm He}$ is the ^{3}He bound state wave function, which we assume to be defined in accordance with the phase convention Eq. (2.26).

We now further simplify this result by employing the Wigner-Eckart theorem. Writing

$$\langle ls; JM | T_{L\lambda} | \phi^\sigma_{\rm He}\rangle = \langle L\lambda, s_c\sigma | JM\rangle \langle ls; J || T_L || \phi_{\rm He}\rangle, \quad (3.19)$$

where s_c is the ^{3}He spin, we obtain

$$T_{\lambda\sigma,\nu_p\nu_d} = -4\pi\left[\frac{\mu_i\omega}{2\pi\hbar}\right.$$

Let us now define a set of three functions, ϕ_α in which the ingoing-wave amplitudes a_β are defined according to the rule

$$a_\beta = u_{\alpha\beta} \tag{4.10}$$

so that

$$\phi_\alpha \rightarrow \sum_\beta \left[u_{\alpha\beta}\chi_\beta^{\rm in} + \sum_\gamma S_{\beta\gamma}u_{\alpha\gamma}\chi_\beta^{\rm out}\right]. \tag{4.11}$$

Then by making use of Eq. (4.4) and by recalling that u is real and unitary and that S_0 is diagonal, one may easily rewrite these functions in the form

$$\phi_\alpha \rightarrow \sum_\beta u_{\alpha\beta}\chi_\beta^{\rm in} + e^{2i\delta_\alpha}\sum_\beta u_{\alpha\beta}\chi_\beta^{\rm out}. \tag{4.12}$$

From this last result one can see why these particular functions are referred to as the eigenstates of the scattering matrix. If we prepare an ingoing-wave state made up of the particular linear combination of partial wave amplitudes specified in Eq. (4.10), then the resulting outgoing wave is just an overall phase, $e^{2i\delta_\alpha}$, times the same linear combination of outgoing partial wave amplitudes.

For our purposes, the importance of this result can be seen by inserting the definitions of χ^{in} and χ^{out} into Eq. (4.12). After some manipulation one obtains

$$\phi_\alpha \rightarrow e^{i\delta_\alpha}\sum_\beta u_{\alpha\beta}\left(\frac{1}{k_i r}\right)\sin\left(k_i r + \delta_\alpha - l_\beta\frac{\pi}{2}\right)\mathcal{Y}^M_{Jl_\beta s_\beta}. \tag{4.13}$$

Recalling once again that the elements of the matrix u are real, we see that the wave functions ϕ_α have a well defined behavior under the operation of time reversal. We introduce states $|\alpha;JM\rangle$ defined by the equation

$$|\alpha;JM\rangle = e^{-i\delta_\alpha}\phi_\alpha, \tag{4.14}$$

where δ_α is the appropriate eigenphaseshift. Then, according to the arguments of Sec. III C, these states must obey the time-reversal transformation

$$\theta_T|\alpha;JM\rangle = (-)^{J-M}|\alpha;J-M\rangle. \tag{4.15}$$

C. The generalized theorem

Since we have now succeeded in identifying states that transform according to the rule (2.26), we can readily obtain the Watson theorem formula for the general case. To do so, we simply need to express the full scattering wave function in terms of the scattering eigenstates.

From Eqs. (3.6) and (3.7) the incident plane wave can be written in the form

$$\psi_{\rm inc} \rightarrow 4\pi \sum_{\substack{J,\pi,\beta \\ M,\nu,m}} \langle s_p\nu_p, s_d\nu_d|s_\beta\nu\rangle\langle l_\beta m, s_\beta\nu|JM\rangle \times(\chi_\beta^{\rm in}+\chi_\beta^{\rm out})Y_{l_\beta}^{m*}(\hat{\mathbf{k}}_i), \tag{4.16}$$

where summation over the β is meant to imply summation over all l-s values associated with each particular J^π. To proceed we now construct the full scattering wave function by making a superposition of scattering eigenstates in which the ingoing waves match those of $\psi_{\rm inc}$. Starting from Eq. (4.12) one readily obtains

$$\sum_\alpha u_{\alpha\beta}\phi_\alpha \rightarrow \chi_\beta^{\rm in} + \sum_\alpha S_{\alpha\beta}\chi_\alpha^{\rm out} \tag{4.17}$$

and so the full elastic scattering wave function can be written in the form

$$\psi^{\nu_p,\nu_d} = 4\pi \sum_{\substack{J,\pi,\beta \\ M,\nu,m}} \langle s_p\nu_p, s_d\nu_d|s_\beta\nu\rangle\langle l_\beta m, s_\beta\nu|JM\rangle \times\left(\sum_\alpha u_{\alpha\beta}\phi_\alpha\right)Y_{l_\beta}^{m*}(\hat{\mathbf{k}}_i). \tag{4.18}$$

The generalized form of Watson's theorem is then obtained by substituting this result into Eq. (2.16). We replace the scattering eigenstates ϕ_α with quantities $|\alpha;JM\rangle$ defined in Eq. (4.14) and thus our expression for the transition amplitude involves matrix elements of the form $\langle\alpha;JM|T_{L\lambda}|\phi_{\rm He}^\sigma\rangle$. We then introduce reduced matrix elements defined by the formula

$$\langle\alpha;JM|T_{L\lambda}|\phi_{\rm He}^\sigma\rangle = \langle L\lambda, s_c\sigma|JM\rangle\langle\alpha;J||T_L||\phi_{\rm He}\rangle. \tag{4.19}$$

Combining the various formulas we then obtain the final result

$$T_{\lambda\sigma,\nu_p\nu_d} = -4\pi\left[\frac{\mu_i\omega}{2\pi\hbar^2 c k_i}\right]^{1/2} \sum_{\substack{J,\pi,\alpha,\beta \\ L,M,\nu,m}} \langle s_p\nu_p, s_d\nu_d|s_\beta\nu\rangle \times\langle l_\beta m, s_\beta\nu|JM\rangle\langle L\lambda, s_c\sigma|JM\rangle \times u_{\alpha\beta}e^{i\delta_\alpha}[\langle\alpha;J||T_L^e||\phi_{\rm He}\rangle^* + \lambda\langle\alpha;J||T_L^m||\phi_{\rm He}\rangle^*]Y_{l_\beta}^{m*}(\hat{\mathbf{k}}_i), \tag{4.20}$$

where, by virtue of Eq. (4.15), it is seen that the reduced matrix elements must once again be real.

V. DISCUSSION

We have now reached the goal of obtaining a single formula which can be used as the basis of a matrix-element analysis of p-d radiative capture data obtained at energies below the deuteron breakup threshold. As we have outlined previously, the approach one uses in such an analysis is to treat the reduced matrix elements as parameters and determine these quantities by fitting measurements.

The advantage of the present formulation over the conventional one is that the matrix elements are necessarily real. Information on the relative phases of the terms that appear in the sum over multipoles and angular momentum states is derived from the elastic scattering channel. The assumption we make is that the eigenphaseshifts and the mixing matrix

elements are known from a separate phase shift analysis of elastic scattering.

In a conventional analysis, the matrix elements appearing in the expansion of T_{fi} would be matrix elements of some multipole operator taken between the bound state and a partial-wave scattering state, $\phi^M_{ls,J}$, defined with the boundary condition that the ingoing asymptotic wave should be a pure angular momentum state involving only a single l-s combination. For these states the outgoing waves have a relatively complicated asymptotic form that, in general, involves all three eigenphaseshifts. As a result the phase of the resulting matrix element is not apparent. In contrast, the matrix elements used in the present formulation are defined in terms of the eigenstates of the S-matrix, which have a simpler asymptotic form that involves only a single eigenphase. Since we use eigenchannel wave functions, we refer to the matrix elements of Eq. (4.19) as eigenchannel matrix elements.

The connection between the eigenchannel matrix elements and the conventional ones is fairly simple. We define a set of matrix element parameters P^{EL}_α and P^{ML}_α by the equations

$$P^{\mathrm{EL}}_\alpha = -\left[\frac{8\mu_i k_i \omega}{\hbar^2 c}\right]^{1/2} \frac{1}{\sqrt{2L+1}} \langle \alpha; J || T^e_L || \phi_{\mathrm{He}} \rangle \qquad (5.1)$$

and

$$P^{\mathrm{ML}}_\alpha = -\left[\frac{8\mu_i k_i \omega}{\hbar^2 c}\right]^{1/2} \frac{1}{\sqrt{2L+1}} \langle \alpha; J || T^m_L || \phi_{\mathrm{He}} \rangle, \qquad (5.2)$$

and then introduce a set of transformed matrix elements,

$$R_\beta = \sum_\alpha u_{\alpha\beta} e^{i\delta_\alpha} P_\alpha . \qquad (5.3)$$

Except for a possible overall phase, these quantities appear to be identical to the R parameters used by Seyler and Weller [1]. In terms of these transformed matrix elements, the transition amplitude may be written as

$$\begin{aligned} T_{\lambda\sigma,\nu_p\nu_d} = \frac{2\pi}{k_i} \sum_{\substack{J,\pi,\beta,L \\ M,\nu,m}} \left[\frac{2L+1}{4\pi}\right]^{1/2} & \langle s_p \nu_p, s_d \nu_d | s_\beta \nu \rangle \\ & \times \langle l_\beta m, s_\beta \nu | JM \rangle \langle L\lambda, s_c \sigma | JM \rangle \\ & \times [R^{\mathrm{EL}}_\beta + \lambda R^{\mathrm{ML}}_\beta] Y^{m*}_{l_\beta}(\hat{\boldsymbol{k}}_i). \end{aligned} \qquad (5.4)$$

While the equations we have obtained for the transition

where W is a diagonal matrix whose elements are the Coulomb phase factors $e^{i\sigma_l}$ associated with the individual angular momentum states that make up S. The phase parameters, σ_l, are obtained in the usual way,

$$\sigma_l = \arg\Gamma(l+1+i\eta). \tag{5.9}$$

It may be noted, however, that one could use modified Coulomb phases

$$\omega_l = \sigma_l - \sigma_0, \tag{5.10}$$

in place of the σ_l's in making the transformation of Eq. (5.8). The only consequence would be that the elements of S would all be rotated by a common phase, $2\sigma_0$, which means that the extracted eigenphases would change by σ_0, and the resulting transition amplitude would thus simply acquire an additional overall phase.

The need to transform from the nuclear S-matrix to the complete S-matrix introduces some complications. In particular one finds that there is no simple relationship between the mixing matrix $\tilde{u}$ that diagonalizes $\tilde{S}$ and the matrix u that diagonalizes S. Similarly the eigenphases of S are not related in any simple way to the eigenphases of $\tilde{S}$. The procedure one might follow to find the needed parameters would be to use the elastic phase shift parameters to construct $\tilde{S}$, find S by the transformation (5.8), and then rediagonalize S to obtain the new eigenphases and mixing matrix elements.

VI. CONCLUSIONS

The use of Watson's theorem for the analysis low-energy radiative capture reactions has been discussed in detail. Particular emphasis has been given to the question of how Watson's theorem can be used in situations in which there is mixing between angular momentum states. Although the present work focuses on the p-d capture reaction, the results obtained here can be applied to other systems as well. The main limitation is that Watson's theorem is valid only in situations in which there are no open reaction channels that significantly reduce the flux in the elastic scattering channel.

It is anticipated that the formalism introduced here will be employed in the near future to carry out a matrix element analysis of a set of p-d radiative capture measurements at $E_{\mathrm{c.m.}}=2$ MeV obtained recently in experiments at the University of Wisconsin [16]. The results of this analysis will be presented in a future publication.

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APPENDIX

For completeness we record here the relevant formulas for the elastic scattering amplitudes. These amplitudes are easily extracted from the scattering wave function given in Eq. (4.18). Using the definition of ϕ_α from Eq. (4.11) one obtains after some algebra

$$\psi^{\nu_p,\nu_d} \rightarrow 4\pi \sum_{\substack{J,l,l',s,s' \\ M,m,\nu}} \langle s_p\nu_p, s_d\nu_d | s\nu\rangle \langle lm, s\nu | JM\rangle \times [\delta_{l,l'}\delta_{s,s'}\chi^{\mathrm{in}}_{Jls} + S^J_{l's',ls}\chi^{\mathrm{out}}_{Jl's'}] Y^{m*}_l(\hat{\boldsymbol{k}}_i). \tag{A1}$$

To find the outgoing wave amplitudes we subtract the incident plane wave, Eq. (3.6). Using the definitions of χ^{in} and χ^{out} from Eqs. (4.7) and (4.8), and inserting the formula for spin-angle functions from Eq. (3.5) one obtains the result

$$\begin{aligned}\psi^{\nu_p,\nu_d}_{\mathrm{scat}} \rightarrow 4\pi \sum_{\substack{J,l,l',s,s' \\ M,m,m',\nu,\nu'}} \sum_{\nu'_p,\nu'_d} & \langle s_p\nu_p, s_d\nu_d | s\nu\rangle \langle lm, s\nu | JM\rangle \\ & \times \langle s_p\nu'_p, s_d\nu'_d | s'\nu'\rangle \langle l'm', s'\nu' | JM\rangle \\ & \times \frac{1}{2i}[S^J_{l's',ls} - \delta_{l,l'}\delta_{s,s'}] \\ & \times Y^{m*}_l(\hat{\boldsymbol{k}}_i) Y^{m'}_{l'}(\hat{\boldsymbol{r}}) \frac{e^{ik_ir}}{k_ir} \phi^{\nu'_p}_p \phi^{\nu'_d}_d .\end{aligned} \tag{A2}$$

One may now read off the outgoing wave amplitude in spin state ν'_p, ν'_d. Adopting the standard coordinate frame for elastic scattering in which the z-axis is along $\boldsymbol{k}_i$ and the y-axis is along $\boldsymbol{k}_i \times \boldsymbol{k}_f$ we have

$$\begin{aligned}M_{\nu'_p\nu'_d;\nu_p\nu_d} = \frac{i\sqrt{\pi}}{k_i} \sum_{\substack{J,l,l',s,s' \\ M,m',\nu,\nu'}} & [2l+1]^{1/2} \langle s_p\nu_p, s_d\nu_d | s\nu\rangle \\ & \times \langle l0, s\nu | JM\rangle \langle s_p\nu'_p, s_d\nu'_d | s'\nu'\rangle \\ & \times \langle l'm', s'\nu' | JM\rangle \\ & \times [\delta_{l,l'}\delta_{s,s'} - S^J_{l's',ls}] Y^{m'}_{l'}(\theta, 0).\end{aligned} \tag{A3}$$

The formulas needed to incorporate the Coulomb scattering in an explicit way are found, for example, in Ref. [17]. By using the analytic expression for the Coulomb amplitude together with the partial wave expansion of the same quantity one obtains the result

$$\begin{aligned}M_{\nu'_p\nu'_d;\nu_p\nu_d} = e^{2i\sigma_0}\frac{\sqrt{\pi}}{k_i}\Bigg\{ & -C(\theta)\delta_{\nu_p,\nu'_p}\delta_{\nu_p,\nu'_p} \\ & + i \sum_{\substack{J,l,l',s,s' \\ M,m',\nu,\nu'}} [2l+1]^{1/2} \langle s_p\nu_p, s_d\nu_d | s\nu\rangle \\ & \times \langle l0, s\nu | JM\rangle \langle s_p\nu'_p, s_d\nu'_d | s'\nu'\rangle\end{aligned}$$

$$\times\langle l'm',s'\nu'|JM\rangle e^{i(\omega_l+\omega_l')}[\delta_{l,l'}\delta_{s,s'} - e^{-i(\sigma_l+\sigma_l')}S^J_{l's',ls}]Y^{m'}_{l'}(\theta,0)\Bigg\}, \tag{A4}$$

where σ_l and ω_l are defined in Eqs. (5.9) and (5.10), respectively, and where $C(\theta)$ is the Coulomb amplitude

$$C(\theta)=\frac{1}{\sqrt{4\pi}}\frac{\eta}{\sin^2\frac{\theta}{2}}e^{-i\eta\ln[\sin^2(\theta/2)]}. \tag{A5}$$

We define the nuclear *S*-matrix elements to be

$$\tilde{S}^J_{l's',ls}=e^{-i(\sigma_l+\sigma_l')}S^J_{l's',ls}, \tag{A6}$$

and in a phase shift analysis the submatrix of $\tilde{S}$ for each J^π value would be parametrized in the form of Eq. (4.4),

$$\tilde{S}=\tilde{u}^\dagger\tilde{S}_0\tilde{u}. \tag{A7}$$

In the matrix element analysis of the capture reaction, one then uses Eq. (5.8) to reconstruct the full *S*-matrix from $\tilde{S}$.

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