

INTEGRALS

ELEMENTARY FORMS

1.  $\int a \, dx = ax$
2.  $\int a \cdot f(x) \, dx = a \int f(x) \, dx$
3.  $\int \phi(y) \, dx = \int \frac{\phi(y)}{y'} \, dy$ , where  $y' = \frac{dy}{dx}$
4.  $\int (u + v) \, dx = \int u \, dx + \int v \, dx$ , where  $u$  and  $v$  are any functions of  $x$
5.  $\int u \, dv = u \int dv - \int v \, du = uv - \int v \, du$
6.  $\int u \frac{dv}{dx} \, dx = uv - \int v \frac{du}{dx} \, dx$
7.  $\int x^n \, dx = \frac{x^{n+1}}{n+1}$ , except  $n = -1$
8.  $\int \frac{f'(x) \, dx}{f(x)} = \log f(x)$ , ( $df(x) = f'(x) \, dx$ )
9.  $\int \frac{dx}{x} = \log x$
10.  $\int \frac{f'(x) \, dx}{2\sqrt{f(x)}} = \sqrt{f(x)}$ , ( $df(x) = f'(x) \, dx$ )
11.  $\int e^x \, dx = e^x$
12.  $\int e^{ax} \, dx = e^{ax}/a$
13.  $\int b^{ax} \, dx = \frac{b^{ax}}{a \log b}$ , ( $b > 0$ )
14.  $\int \log x \, dx = x \log x - x$
15.  $\int a^x \log a \, dx = a^x$ , ( $a > 0$ )
16.  $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$
17.  $\int \frac{dx}{a^2 - x^2} = \begin{cases} \frac{1}{a} \tanh^{-1} \frac{x}{a} \\ \text{or} \\ \frac{1}{2a} \log \frac{a+x}{a-x}, \end{cases} \quad (a^2 > x^2)$
18.  $\int \frac{dx}{x^2 - a^2} = \begin{cases} -\frac{1}{a} \coth^{-1} \frac{x}{a} \\ \text{or} \\ \frac{1}{2a} \log \frac{x-a}{x+a}, \end{cases} \quad (x^2 > a^2)$

INTEGRALS (Continued)

$$19. \int \frac{dx}{\sqrt{a^2 - x^2}} = \begin{cases} \sin^{-1} \frac{x}{|a|} \\ \text{or} \\ -\cos^{-1} \frac{x}{|a|}, \end{cases} \quad (a^2 > x^2)$$

$$20. \int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log(x + \sqrt{x^2 \pm a^2})$$

$$21. \int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{|a|} \sec^{-1} \frac{x}{a}$$

$$22. \int \frac{dx}{x\sqrt{a^2 \pm x^2}} = -\frac{1}{a} \log \left( \frac{a + \sqrt{a^2 \pm x^2}}{x} \right)$$

FORMS CONTAINING  $(a + bx)$

For forms containing  $a + bx$ , but not listed in the table, the substitution  $u = \frac{a + bx}{x}$  may prove helpful.

$$23. \int (a + bx)^n dx = \frac{(a + bx)^{n+1}}{(n+1)b}, \quad (n \neq -1)$$

$$24. \int x(a + bx)^n dx = \frac{1}{b^2(n+2)}(a + bx)^{n+2} - \frac{a}{b^2(n+1)}(a + bx)^{n+1}, \quad (n \neq -1, -2)$$

$$25. \int x^2(a + bx)^n dx = \frac{1}{b^3} \left[ \frac{(a + bx)^{n+3}}{n+3} - 2a \frac{(a + bx)^{n+2}}{n+2} + a^2 \frac{(a + bx)^{n+1}}{n+1} \right]$$

$$26. \int x^m(a + bx)^n dx = \begin{cases} \frac{x^{m+1}(a + bx)^n}{m+n+1} + \frac{an}{m+n+1} \int x^m(a + bx)^{n-1} dx \\ \text{or} \\ \frac{1}{a(n+1)} \left[ -x^{m+1}(a + bx)^{n+1} + (m+n+2) \int x^m(a + bx)^{n+1} dx \right] \\ \text{or} \\ \frac{1}{b(m+n+1)} \left[ x^m(a + bx)^{n+1} - ma \int x^{m-1}(a + bx)^n dx \right] \end{cases}$$

$$27. \int \frac{dx}{a + bx} = \frac{1}{b} \log(a + bx)$$

$$28. \int \frac{dx}{(a + bx)^2} = -\frac{1}{b(a + bx)}$$

$$29. \int \frac{dx}{(a + bx)^3} = -\frac{1}{2b(a + bx)^2}$$

$$30. \int \frac{x dx}{a + bx} = \begin{cases} \frac{1}{b^2} [a + bx - a \log(a + bx)] \\ \text{or} \\ \frac{x}{b} - \frac{a}{b^2} \log(a + bx) \end{cases}$$

INTEGRALS (Continued)

31.  $\int \frac{x dx}{(a + bx)^2} = \frac{1}{b^2} \left[ \log(a + bx) + \frac{a}{a + bx} \right]$
32.  $\int \frac{x dx}{(a + bx)^n} = \frac{1}{b^2} \left[ \frac{-1}{(n-2)(a + bx)^{n-2}} + \frac{a}{(n-1)(a + bx)^{n-1}} \right], \quad n \neq 1, 2$
33.  $\int \frac{x^2 dx}{a + bx} = \frac{1}{b^3} \left[ \frac{1}{2}(a + bx)^2 - 2a(a + bx) + a^2 \log(a + bx) \right]$
34.  $\int \frac{x^2 dx}{(a + bx)^2} = \frac{1}{b^3} \left[ a + bx - 2a \log(a + bx) - \frac{a^2}{a + bx} \right]$
35.  $\int \frac{x^2 dx}{(a + bx)^3} = \frac{1}{b^3} \left[ \log(a + bx) + \frac{2a}{a + bx} - \frac{a^2}{2(a + bx)^2} \right]$
36.  $\int \frac{x^2 dx}{(a + bx)^n} = \frac{1}{b^3} \left[ \frac{-1}{(n-3)(a + bx)^{n-3}} + \frac{2a}{(n-2)(a + bx)^{n-2}} - \frac{a^2}{(n-1)(a + bx)^{n-1}} \right], \quad n \neq 1, 2, 3$
37.  $\int \frac{dx}{x(a + bx)} = -\frac{1}{a} \log \frac{a + bx}{x}$
38.  $\int \frac{dx}{x(a + bx)^2} = \frac{1}{a(a + bx)} - \frac{1}{a^2} \log \frac{a + bx}{x}$
39.  $\int \frac{dx}{x(a + bx)^3} = \frac{1}{a^3} \left[ \frac{1}{2} \left( \frac{2a + bx}{a + bx} \right)^2 + \log \frac{x}{a + bx} \right]$
40.  $\int \frac{dx}{x^2(a + bx)} = -\frac{1}{ax} + \frac{b}{a^2} \log \frac{a + bx}{x}$
41.  $\int \frac{dx}{x^3(a + bx)} = \frac{2bx - a}{2a^2x^2} + \frac{b^2}{a^3} \log \frac{x}{a + bx}$
42.  $\int \frac{dx}{x^2(a + bx)^2} = -\frac{a + 2bx}{a^2x(a + bx)} + \frac{2b}{a^3} \log \frac{a + bx}{x}$

FORMS CONTAINING  $c^2 \pm x^2, x^2 - c^2$

43.  $\int \frac{dx}{c^2 + x^2} = \frac{1}{c} \tan^{-1} \frac{x}{c}$
44.  $\int \frac{dx}{c^2 - x^2} = \frac{1}{2c} \log \frac{c + x}{c - x}, \quad (c^2 > x^2)$
45.  $\int \frac{dx}{x^2 - c^2} = \frac{1}{2c} \log \frac{x - c}{x + c}, \quad (x^2 > c^2)$
46.  $\int \frac{x dx}{c^2 \pm x^2} = \pm \frac{1}{2} \log(c^2 \pm x^2)$
47.  $\int \frac{x dx}{(c^2 \pm x^2)^{n+1}} = \mp \frac{1}{2n(c^2 \pm x^2)^n}$
48.  $\int \frac{dx}{(c^2 \pm x^2)^n} = \frac{1}{2c^2(n-1)} \left[ \frac{x}{(c^2 \pm x^2)^{n-1}} + (2n-3) \int \frac{dx}{(c^2 \pm x^2)^{n-1}} \right]$
49.  $\int \frac{dx}{(x^2 - c^2)^n} = \frac{1}{2c^2(n-1)} \left[ -\frac{x}{(x^2 - c^2)^{n-1}} - (2n-3) \int \frac{dx}{(x^2 - c^2)^{n-1}} \right]$
50.  $\int \frac{x dx}{x^2 - c^2} = \frac{1}{2} \log(x^2 - c^2)$

INTEGRALS (Continued)

$$51. \int \frac{x dx}{(x^2 - c^2)^{n+1}} = -\frac{1}{2n(x^2 - c^2)^n}$$

FORMS CONTAINING  $a + bx$  and  $c + dx$

$$u = a + bx, \quad v = c + dx, \quad k = ad - bc$$

If  $k = 0$ , then  $v = \frac{c}{a}u$

$$52. \int \frac{dx}{u \cdot v} = \frac{1}{k} \cdot \log\left(\frac{v}{u}\right)$$

$$53. \int \frac{x dx}{u \cdot v} = \frac{1}{k} \left[ \frac{a}{b} \log(u) - \frac{c}{d} \log(v) \right]$$

$$54. \int \frac{dx}{u^2 \cdot v} = \frac{1}{k} \left( \frac{1}{u} + \frac{d}{k} \log \frac{v}{u} \right)$$

$$55. \int \frac{x dx}{u^2 \cdot v} = \frac{-a}{bku} - \frac{c}{k^2} \log \frac{v}{u}$$

$$56. \int \frac{x^2 dx}{u^2 \cdot v} = \frac{a^2}{b^2ku} + \frac{1}{k^2} \left[ \frac{c^2}{d} \log(v) + \frac{a(k-bc)}{b^2} \log(u) \right]$$

$$57. \int \frac{dx}{u^m \cdot v^n} = \frac{1}{k(m-1)} \left[ \frac{-1}{u^{m-1} \cdot v^{n-1}} - (m+n-2)b \int \frac{dx}{u^m \cdot v^{n-1}} \right]$$

$$58. \int \frac{u}{v} dx = \frac{bx}{d} + \frac{k}{d^2} \log(v)$$

$$59. \int \frac{u^m dx}{v^n} = \begin{cases} \frac{-1}{k(n-1)} \left[ \frac{u^{m+1}}{v^{n-1}} + b(n-m-2) \int \frac{u^m}{v^{n-1}} dx \right] \\ \text{or} \\ \frac{-1}{d(n-m-1)} \left[ \frac{u^m}{v^{n-1}} + mk \int \frac{u^{m-1}}{v^n} dx \right] \\ \text{or} \\ \frac{-1}{d(n-1)} \left[ \frac{u^m}{v^{n-1}} - mb \int \frac{u^{m-1}}{v^{n-1}} dx \right] \end{cases}$$

FORMS CONTAINING  $(a + bx^n)$

$$60. \int \frac{dx}{a + bx^2} = \frac{1}{\sqrt{ab}} \tan^{-1} \frac{x\sqrt{ab}}{a}, \quad (ab > 0)$$

$$61. \int \frac{dx}{a + bx^2} = \begin{cases} \frac{1}{2\sqrt{-ab}} \log \frac{a + x\sqrt{-ab}}{a - x\sqrt{-ab}}, & (ab < 0) \\ \text{or} \\ \frac{1}{\sqrt{-ab}} \tanh^{-1} \frac{x\sqrt{-ab}}{a}, & (ab < 0) \end{cases}$$

$$62. \int \frac{dx}{a^2 + b^2x^2} = \frac{1}{ab} \tan^{-1} \frac{bx}{a}$$

$$63. \int \frac{x dx}{a + bx^2} = \frac{1}{2b} \log(a + bx^2)$$

$$64. \int \frac{x^2 dx}{a + bx^2} = \frac{x}{b} - \frac{a}{b} \int \frac{dx}{a + bx^2}$$

$$65. \int \frac{dx}{(a + bx^2)^2} = \frac{x}{2a(a + bx^2)} + \frac{1}{2a} \int \frac{dx}{a + bx^2}$$

INTEGRALS (Continued)

$$66. \int \frac{dx}{a^2 - b^2x^2} = \frac{1}{2ab} \log \frac{a + bx}{a - bx}$$

$$67. \int \frac{dx}{(a + bx^2)^{m+1}} = \begin{cases} \frac{1}{2ma(a + bx^2)^m} + \frac{2m-1}{2ma} \int \frac{dx}{(a + bx^2)^m} \\ \text{or} \\ \frac{(2m)!}{(m!)^2} \left[ \frac{x}{2a} \sum_{r=1}^m \frac{r!(r-1)!}{(4a)^{m-r}(2r)!(a + bx^2)^r} + \frac{1}{(4a)^m} \int \frac{dx}{a + bx^2} \right] \end{cases}$$

$$68. \int \frac{x dx}{(a + bx^2)^{m+1}} = -\frac{1}{2bm(a + bx^2)^m}$$

$$69. \int \frac{x^2 dx}{(a + bx^2)^{m+1}} = \frac{-x}{2mb(a + bx^2)^m} + \frac{1}{2mb} \int \frac{dx}{(a + bx^2)^m}$$

$$70. \int \frac{dx}{x(a + bx^2)} = \frac{1}{2a} \log \frac{x^2}{a + bx^2}$$

$$71. \int \frac{dx}{x^2(a + bx^2)} = -\frac{1}{ax} - \frac{b}{a} \int \frac{dx}{a + bx^2}$$

$$72. \int \frac{dx}{x(a + bx^2)^{m+1}} = \begin{cases} \frac{1}{2am(a + bx^2)^m} + \frac{1}{a} \int \frac{dx}{x(a + bx^2)^m} \\ \text{or} \\ \frac{1}{2a^{m+1}} \left[ \sum_{r=1}^m \frac{a^r}{r(a + bx^2)^r} + \log \frac{x^2}{a + bx^2} \right] \end{cases}$$

$$73. \int \frac{dx}{x^2(a + bx^2)^{m+1}} = \frac{1}{a} \int \frac{dx}{x^2(a + bx^2)^m} - \frac{b}{a} \int \frac{dx}{(a + bx^2)^{m+1}}$$

$$74. \int \frac{dx}{a + bx^3} = \frac{k}{3a} \left[ \frac{1}{2} \log \frac{(k+x)^3}{a + bx^3} + \sqrt{3} \tan^{-1} \frac{2x-k}{k\sqrt{3}} \right], \quad \left( k = \sqrt[3]{\frac{a}{b}} \right)$$

$$75. \int \frac{x dx}{a + bx^3} = \frac{1}{3bk} \left[ \frac{1}{2} \log \frac{a + bx^3}{(k+x)^3} + \sqrt{3} \tan^{-1} \frac{2x-k}{k\sqrt{3}} \right], \quad \left( k = \sqrt[3]{\frac{a}{b}} \right)$$

$$76. \int \frac{x^2 dx}{a + bx^3} = \frac{1}{3b} \log(a + bx^3)$$

$$77. \int \frac{dx}{a + bx^4} = \frac{k}{2a} \left[ \frac{1}{2} \log \frac{x^2 + 2kx + 2k^2}{x^2 - 2kx + 2k^2} + \tan^{-1} \frac{2kx}{2k^2 - x^2} \right], \quad \left( ab > 0, k = \sqrt[4]{\frac{a}{4b}} \right)$$

$$78. \int \frac{dx}{a + bx^4} = \frac{k}{2a} \left[ \frac{1}{2} \log \frac{x+k}{x-k} + \tan^{-1} \frac{x}{k} \right], \quad \left( ab < 0, k = \sqrt[4]{-\frac{a}{b}} \right)$$

$$79. \int \frac{x dx}{a + bx^4} = \frac{1}{2bk} \tan^{-1} \frac{x^2}{k}, \quad \left( ab > 0, k = \sqrt{\frac{a}{b}} \right)$$

$$80. \int \frac{x dx}{a + bx^4} = \frac{1}{4bk} \log \frac{x^2 - k}{x^2 + k}, \quad \left( ab < 0, k = \sqrt{-\frac{a}{b}} \right)$$

$$81. \int \frac{x^2 dx}{a + bx^4} = \frac{1}{4bk} \left[ \frac{1}{2} \log \frac{x^2 - 2kx + 2k^2}{x^2 + 2kx + 2k^2} + \tan^{-1} \frac{2kx}{2k^2 - x^2} \right], \quad \left( ab > 0, k = \sqrt[4]{\frac{a}{4b}} \right)$$

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$$82. \int \frac{x^2 dx}{a + bx^4} = \frac{1}{4bk} \left[ \log \frac{x-k}{x+k} + 2 \tan^{-1} \frac{x}{k} \right], \quad (ab < 0, k = \sqrt[4]{-\frac{a}{b}})$$

$$83. \int \frac{x^3 dx}{a + bx^4} = \frac{1}{4b} \log(a + bx^4)$$

$$84. \int \frac{dx}{x(a + bx^n)} = \frac{1}{an} \log \frac{x^n}{a + bx^n}$$

$$85. \int \frac{dx}{(a + bx^n)^{m+1}} = \frac{1}{a} \int \frac{dx}{(a + bx^n)^m} - \frac{b}{a} \int \frac{x^n dx}{(a + bx^n)^{m+1}}$$

$$86. \int \frac{x^n dx}{(a + bx^n)^{p+1}} = \frac{1}{b} \int \frac{x^{m-n} dx}{(a + bx^n)^p} - \frac{a}{b} \int \frac{x^{m-n} dx}{(a + bx^n)^{p+1}}$$

$$87. \int \frac{dx}{x^m(a + bx^n)^{p+1}} = \frac{1}{a} \int \frac{dx}{x^m(a + bx^n)^p} - \frac{b}{a} \int \frac{dx}{x^{m-n}(a + bx^n)^{p+1}}$$

$$88. \int x^m(a + bx^n)^p dx = \begin{cases} \frac{1}{b(np + m + 1)} \left[ x^{m-n+1}(a + bx^n)^{p+1} - a(m - n + 1) \int x^{m-n}(a + bx^n)^p dx \right] \\ \text{or} \\ \frac{1}{np + m + 1} \left[ x^{m+1}(a + bx^n)^p + anp \int x^m(a + bx^n)^{p-1} dx \right] \\ \text{or} \\ \frac{1}{a(m + 1)} \left[ x^{m+1}(a + bx^n)^{p+1} - (m + 1 + np + n)b \int x^{m+n}(a + bx^n)^p dx \right] \\ \text{or} \\ \frac{1}{an(p + 1)} \left[ -x^{m+1}(a + bx^n)^{p+1} + (m + 1 + np + n) \int x^m(a + bx^n)^{p+1} dx \right] \end{cases}$$

FORMS CONTAINING  $c^3 \pm x^3$

$$89. \int \frac{dx}{c^3 \pm x^3} = \pm \frac{1}{6c^2} \log \frac{(c \pm x)^3}{c^3 \pm x^3} + \frac{1}{c^2\sqrt{3}} \tan^{-1} \frac{2x \mp c}{c\sqrt{3}}$$

$$90. \int \frac{dx}{(c^3 \pm x^3)^2} = \frac{x}{3c^3(c^3 \pm x^3)} + \frac{2}{3c^3} \int \frac{dx}{c^3 \pm x^3}$$

$$91. \int \frac{dx}{(c^3 \pm x^3)^{n+1}} = \frac{1}{3nc^3} \left[ \frac{x}{(c^3 \pm x^3)^n} + (3n - 1) \int \frac{dx}{(c^3 \pm x^3)^n} \right]$$

$$92. \int \frac{x dx}{c^3 \pm x^3} = \frac{1}{6c} \log \frac{c^3 \pm x^3}{(c \pm x)^3} \pm \frac{1}{c\sqrt{3}} \tan^{-1} \frac{2x \mp c}{c\sqrt{3}}$$

$$93. \int \frac{x dx}{(c^3 \pm x^3)^2} = \frac{x^2}{3c^3(c^3 \pm x^3)} + \frac{1}{3c^3} \int \frac{x dx}{c^3 \pm x^3}$$

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94.  $\int \frac{x dx}{(c^3 \pm x^3)^{n+1}} = \frac{1}{3nc^3} \left[ \frac{x^2}{(c^3 \pm x^3)^n} + (3n-2) \int \frac{x dx}{(c^3 \pm x^3)^n} \right]$
95.  $\int \frac{x^2 dx}{c^3 \pm x^3} = \pm \frac{1}{3} \log(c^3 \pm x^3)$
96.  $\int \frac{x^2 dx}{(c^3 \pm x^3)^{n+1}} = \mp \frac{1}{3n(c^3 \pm x^3)^n}$
97.  $\int \frac{dx}{x(c^3 \pm x^3)} = \frac{1}{3c^3} \log \frac{x^3}{c^3 \pm x^3}$
98.  $\int \frac{dx}{x(c^3 \pm x^3)^2} = \frac{1}{3c^3(c^3 \pm x^3)} + \frac{1}{3c^6} \log \frac{x^3}{c^3 \pm x^3}$
99.  $\int \frac{dx}{x(c^3 \pm x^3)^{n+1}} = \frac{1}{3nc^3(c^3 \pm x^3)^n} + \frac{1}{c^3} \int \frac{dx}{x(c^3 \pm x^3)^n}$
100.  $\int \frac{dx}{x^2(c^3 \pm x^3)} = -\frac{1}{c^3 x} \mp \frac{1}{c^3} \int \frac{x dx}{c^3 \pm x^3}$
101.  $\int \frac{dx}{x^2(c^3 \pm x^3)^{n+1}} = \frac{1}{c^3} \int \frac{dx}{x^2(c^3 \pm x^3)^n} \mp \frac{1}{c^3} \int \frac{x dx}{(c^3 \pm x^3)^{n+1}}$

FORMS CONTAINING  $c^4 \pm x^4$

102.  $\int \frac{dx}{c^4 + x^4} = \frac{1}{2c^3\sqrt{2}} \left[ \frac{1}{2} \log \frac{x^2 + cx\sqrt{2} + c^2}{x^2 - cx\sqrt{2} + c^2} + \tan^{-1} \frac{cx\sqrt{2}}{c^2 - x^2} \right]$
103.  $\int \frac{dx}{c^4 - x^4} = \frac{1}{2c^3} \left[ \frac{1}{2} \log \frac{c+x}{c-x} + \tan^{-1} \frac{x}{c} \right]$
104.  $\int \frac{x dx}{c^4 + x^4} = \frac{1}{2c^2} \tan^{-1} \frac{x^2}{c^2}$
105.  $\int \frac{x dx}{c^4 - x^4} = \frac{1}{4c^2} \log \frac{c^2 + x^2}{c^2 - x^2}$
106.  $\int \frac{x^2 dx}{c^4 + x^4} = \frac{1}{2c\sqrt{2}} \left[ \frac{1}{2} \log \frac{x^2 - cx\sqrt{2} + c^2}{x^2 + cx\sqrt{2} + c^2} + \tan^{-1} \frac{cx\sqrt{2}}{c^2 - x^2} \right]$
107.  $\int \frac{x^2 dx}{c^4 - x^4} = \frac{1}{2c} \left[ \frac{1}{2} \log \frac{c+x}{c-x} - \tan^{-1} \frac{x}{c} \right]$
108.  $\int \frac{x^3 dx}{c^4 \pm x^4} = \pm \frac{1}{4} \log(c^4 \pm x^4)$

FORMS CONTAINING  $(a+bx+cx^2)$

$$X = a + bx + cx^2 \text{ and } q = 4ac - b^2$$

If  $q=0$ , then  $X = c\left(x + \frac{b}{2c}\right)^2$ , and formulas starting with 23 should be used in place of these.

109.  $\int \frac{dx}{X} = \frac{2}{\sqrt{q}} \tan^{-1} \frac{2cx+b}{\sqrt{q}}, \quad (q > 0)$

110.  $\int \frac{dx}{X} = \begin{cases} \frac{-2}{\sqrt{-q}} \tanh^{-1} \frac{2cx+b}{\sqrt{-q}} \\ \text{or} \\ \frac{1}{\sqrt{-q}} \log \frac{2cx+b-\sqrt{-q}}{2cx+b+\sqrt{-q}}, \end{cases} \quad (q < 0)$

111.  $\int \frac{dx}{X^2} = \frac{2cx+b}{qX} + \frac{2c}{q} \int \frac{dx}{X}$

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$$112. \int \frac{dx}{X^3} = \frac{2cx + b}{q} \left( \frac{1}{2X^2} + \frac{3c}{qX} \right) + \frac{6c^2}{q^2} \int \frac{dx}{X}$$

$$113. \int \frac{dx}{X^{n+1}} = \begin{cases} \frac{2cx + b}{nqX^n} + \frac{2(2n-1)c}{qn} \int \frac{dx}{X^n} \\ \text{or} \\ \frac{(2n)!}{(n!)^2} \left( \frac{c}{q} \right)^n \left[ \frac{2cx + b}{q} \sum_{r=1}^n \left( \frac{q}{cX} \right)^r \frac{(r-1)!r!}{(2r)!} \right] + \int \frac{dx}{X} \end{cases}$$

$$114. \int \frac{x dx}{X} = \frac{1}{2c} \log X - \frac{b}{2c} \int \frac{dx}{X}$$

$$115. \int \frac{x dx}{X^2} = \frac{bx + 2a}{qX} - \frac{b}{q} \int \frac{dx}{X}$$

$$116. \int \frac{x dx}{X^{n+1}} = -\frac{2a + bx}{nqX^n} - \frac{b(2n-1)}{nq} \int \frac{dx}{X^n}$$

$$117. \int \frac{x^2 dx}{X} = \frac{x}{c} - \frac{b}{2c^2} \log X + \frac{b^2 - 2ac}{2c^2} \int \frac{dx}{X}$$

$$118. \int \frac{x^2 dx}{X^2} = \frac{(b^2 - 2ac)x + ab}{cqX} + \frac{2a}{q} \int \frac{dx}{X}$$

$$119. \int \frac{x^m dx}{X^{n+1}} = -\frac{x^{m-1}}{(2n-m+1)cX^n} - \frac{n-m+1}{2n-m+1} \cdot \frac{b}{c} \int \frac{x^{m-1} dx}{X^{n+1}} \\ + \frac{m-1}{2n-m+1} \cdot \frac{a}{c} \int \frac{x^{m-2} dx}{X^{n+1}}$$

$$120. \int \frac{dx}{xX} = \frac{1}{2a} \log \frac{x^2}{X} - \frac{b}{2a} \int \frac{dx}{X}$$

$$121. \int \frac{dx}{x^2X} = \frac{b}{2a^2} \log \frac{X}{x^2} - \frac{1}{ax} + \left( \frac{b^2}{2a^2} - \frac{c}{a} \right) \int \frac{dx}{X}$$

$$122. \int \frac{dx}{xX^n} = \frac{1}{2a(n-1)X^{n-1}} - \frac{b}{2a} \int \frac{dx}{X^n} + \frac{1}{a} \int \frac{dx}{xX^{n-1}}$$

$$123. \int \frac{dx}{x^mX^{n+1}} = -\frac{1}{(m-1)ax^{m-1}X^n} - \frac{n+m-1}{m-1} \cdot \frac{b}{a} \int \frac{dx}{x^{m-1}X^{n+1}} \\ - \frac{2n+m-1}{m-1} \cdot \frac{c}{a} \int \frac{dx}{x^{m-2}X^{n+1}}$$

FORMS CONTAINING  $\sqrt{a+bx}$

$$124. \int \sqrt{a+bx} dx = \frac{2}{3b} \sqrt{(a+bx)^3}$$

$$125. \int x\sqrt{a+bx} dx = -\frac{2(2a-3bx)\sqrt{(a+bx)^3}}{15b^2}$$

$$126. \int x^2\sqrt{a+bx} dx = \frac{2(8a^2 - 12abx + 15b^2x^2)\sqrt{(a+bx)^3}}{105b^3}$$

$$127. \int x^m \sqrt{a+bx} dx = \begin{cases} \frac{2}{b(2m+3)} \left[ x^m \sqrt{(a+bx)^3} - ma \int x^{m-1} \sqrt{a+bx} dx \right] \\ \text{or} \\ \frac{2}{b^{m+1}} \sqrt{a+bx} \sum_{r=0}^m \frac{m!(-a)^{m-r}}{r!(m-r)!(2r+3)} (a+bx)^{r+1} \end{cases}$$



INTEGRALS (Continued)

128.  $\int \frac{\sqrt{a+bx}}{x} dx = 2\sqrt{a+bx} + a \int \frac{dx}{x\sqrt{a+bx}}$
129.  $\int \frac{\sqrt{a+bx}}{x^2} dx = -\frac{\sqrt{a+bx}}{x} + \frac{b}{2} \int \frac{dx}{x\sqrt{a+bx}}$
130.  $\int \frac{\sqrt{a+bx}}{x^m} dx = -\frac{1}{(m-1)a} \left[ \frac{\sqrt{(a+bx)^3}}{x^{m-1}} + \frac{(2m-5)b}{2} \int \frac{\sqrt{a+bx}}{x^{m-1}} dx \right]$
131.  $\int \frac{dx}{\sqrt{a+bx}} = \frac{2\sqrt{a+bx}}{b}$
132.  $\int \frac{x dx}{\sqrt{a+bx}} = -\frac{2(2a-bx)}{3b^2} \sqrt{a+bx}$
133.  $\int \frac{x^2 dx}{\sqrt{a+bx}} = \frac{2(8a^2 - 4abx - 3b^2x^2)}{15b^3} \sqrt{a+bx}$
134.  $\int \frac{x^m dx}{\sqrt{a+bx}} = \begin{cases} \frac{2}{(2m+1)b} \left[ x^m \sqrt{a+bx} - ma \int \frac{x^{m-1} dx}{\sqrt{a+bx}} \right] \\ \text{or} \\ \frac{2(-a)^m \sqrt{a+bx}}{b^{m+1}} \sum_{r=0}^m \frac{(-1)^r m!(a+bx)^r}{(2r+1)r!(m-r)! a^r} \end{cases}$
135.  $\int \frac{dx}{x\sqrt{a+bx}} = \frac{1}{\sqrt{a}} \log \left( \frac{\sqrt{a+bx} - \sqrt{a}}{\sqrt{a+bx} + \sqrt{a}} \right), \quad (a > 0)$
136.  $\int \frac{dx}{x\sqrt{a+bx}} = \frac{2}{\sqrt{-a}} \tan^{-1} \sqrt{\frac{a+bx}{-a}}, \quad (a < 0)$
137.  $\int \frac{dx}{x^2\sqrt{a+bx}} = -\frac{\sqrt{a+bx}}{ax} - \frac{b}{2a} \int \frac{dx}{x\sqrt{a+bx}}$
138.  $\int \frac{dx}{x^n\sqrt{a+bx}} = \begin{cases} -\frac{\sqrt{a+bx}}{(n-1)ax^{n-1}} - \frac{(2n-3)b}{(2n-2)a} \int \frac{dx}{x^{n-1}\sqrt{a+bx}} \\ \frac{(2n-2)!}{[(n-1)!]^2} \left[ -\frac{\sqrt{a+bx}}{a} \sum_{r=1}^{n-1} \frac{r!(r-1)!}{x^r 2(r)!} \left( -\frac{b}{4a} \right)^{n-r-1} \right. \\ \left. + \left( -\frac{b}{4a} \right)^{n-1} \int \frac{dx}{x\sqrt{a+bx}} \right] \end{cases}$
139.  $\int (a+bx)^{\pm \frac{n}{2}} dx = \frac{2(a+bx)^{\frac{2\pm n}{2}}}{b(2\pm n)}$
140.  $\int x(a+bx)^{\pm \frac{n}{2}} dx = \frac{2}{b^2} \left[ \frac{(a+bx)^{\frac{4\pm n}{2}}}{4\pm n} - \frac{a(a+bx)^{\frac{2\pm n}{2}}}{2\pm n} \right]$
141.  $\int \frac{dx}{x(a+bx)^{\frac{m}{2}}} = \frac{1}{a} \int \frac{dx}{x(a+bx)^{\frac{m-2}{2}}} - \frac{b}{a} \int \frac{dx}{(a+bx)^{\frac{m}{2}}}$

INTEGRALS (Continued)

$$142. \int \frac{(a+bx)^{n/2} dx}{x} = b \int (a+bx)^{(n-2)/2} dx + a \int \frac{(a+bx)^{(n-2)/2}}{x} dx$$

$$143. \int f(x, \sqrt{a+bx}) dx = \frac{2}{b} \int f\left(\frac{z^2-a}{b}, z\right) z dz, \quad (z = \sqrt{a+bx})$$

FORMS CONTAINING  $\sqrt{a+bx}$  and  $\sqrt{c+dx}$

$$u = a+bx \quad v = c+dx \quad k = ad-bc$$

If  $k=0$ , then,  $v = \frac{c}{a}u$ , and formulas starting with 124 should be used in place of these.

$$144. \int \frac{dx}{\sqrt{uv}} = \begin{cases} \frac{2}{\sqrt{bd}} \tanh^{-1} \frac{\sqrt{bd}uv}{bv}, & bd > 0, k < 0 \\ \text{or} \\ \frac{2}{\sqrt{bd}} \tanh^{-1} \frac{\sqrt{bd}uv}{du}, & bd > 0, k > 0. \\ \text{or} \\ \frac{1}{\sqrt{bd}} \log \frac{(bv + \sqrt{bd}uv)^2}{v}, & (bd > 0) \end{cases}$$

$$145. \int \frac{dx}{\sqrt{uv}} = \begin{cases} \frac{2}{\sqrt{-bd}} \tan^{-1} \frac{\sqrt{-bd}uv}{bv} \\ \text{or} \\ -\frac{1}{\sqrt{-bd}} \sin^{-1} \left( \frac{2bdx + ad + bc}{|k|} \right), & (bd < 0) \end{cases}$$

$$146. \int \sqrt{uv} dx = \frac{k+2bv}{4bd} \sqrt{uv} - \frac{k^2}{8bd} \int \frac{dx}{\sqrt{uv}}$$

$$147. \int \frac{dx}{v\sqrt{u}} = \begin{cases} \frac{1}{\sqrt{kd}} \log \frac{d\sqrt{u} - \sqrt{kd}}{d\sqrt{u} + \sqrt{kd}} \\ \text{or} \\ \frac{1}{\sqrt{kd}} \log \frac{(d\sqrt{u} - \sqrt{kd})^2}{v}, & (kd > 0) \end{cases}$$

$$148. \int \frac{dx}{v\sqrt{u}} = \frac{2}{\sqrt{-kd}} \tan^{-1} \frac{d\sqrt{u}}{\sqrt{-kd}}, \quad (kd < 0)$$

$$149. \int \frac{x dx}{\sqrt{uv}} = \frac{\sqrt{uv}}{bd} - \frac{ad+bc}{2bd} \int \frac{dx}{\sqrt{uv}}$$

$$150. \int \frac{dx}{v\sqrt{uv}} = \frac{-2\sqrt{uv}}{kv}$$

$$151. \int \frac{v dx}{\sqrt{uv}} = \frac{\sqrt{uv}}{b} - \frac{k}{2b} \int \frac{dx}{\sqrt{uv}}$$

$$152. \int \frac{\sqrt{v}}{\sqrt{u}} dx = \frac{v}{|v|} \int \frac{v dx}{\sqrt{uv}}$$

$$153. \int v^m \sqrt{u} dx = \frac{1}{(2m+3)d} \left( 2v^{m+1} \sqrt{u} + k \int \frac{v^m dx}{\sqrt{u}} \right)$$

INTEGRALS (Continued)

$$154. \int \frac{dx}{v^m \sqrt{u}} = -\frac{1}{(m-1)k} \left( \frac{\sqrt{u}}{v^{m-1}} + \left(m - \frac{3}{2}\right)b \int \frac{dx}{v^{m-1} \sqrt{u}} \right)$$

$$155. \int \frac{v^m dx}{\sqrt{u}} = \begin{cases} \frac{2}{b(2m+1)} \left[ v^m \sqrt{u} - mk \int \frac{v^{m-1}}{\sqrt{u}} dx \right] \\ \text{or} \\ \frac{2(m!)^2 \sqrt{u}}{b(2m+1)!} \sum_{r=0}^m \left(-\frac{4k}{b}\right)^{m-r} \frac{(2r)!}{(r!)^2} v^r \end{cases}$$

FORMS CONTAINING  $\sqrt{x^2 \pm a^2}$

$$156. \int \sqrt{x^2 \pm a^2} dx = \frac{1}{2} [x\sqrt{x^2 \pm a^2} \pm a^2 \log(x + \sqrt{x^2 \pm a^2})]$$

$$157. \int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log(x + \sqrt{x^2 \pm a^2})$$

$$158. \int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{|a|} \sec^{-1} \frac{x}{a}$$

$$159. \int \frac{dx}{x\sqrt{x^2 + a^2}} = -\frac{1}{a} \log\left(\frac{a + \sqrt{x^2 + a^2}}{x}\right)$$

$$160. \int \frac{\sqrt{x^2 + a^2}}{x} dx = \sqrt{x^2 + a^2} - a \log\left(\frac{a + \sqrt{x^2 + a^2}}{x}\right)$$

$$161. \int \frac{\sqrt{x^2 - a^2}}{x} dx = \sqrt{x^2 - a^2} - |a| \sec^{-1} \frac{x}{a}$$

$$162. \int \frac{x dx}{\sqrt{x^2 \pm a^2}} = \sqrt{x^2 \pm a^2}$$

$$163. \int x\sqrt{x^2 \pm a^2} dx = \frac{1}{3} \sqrt{(x^2 \pm a^2)^3}$$

$$164. \int \sqrt{(x^2 \pm a^2)^3} dx = \frac{1}{4} \left[ x\sqrt{(x^2 \pm a^2)^3} \pm \frac{3a^2 x}{2} \sqrt{x^2 \pm a^2} + \frac{3a^4}{2} \log(x + \sqrt{x^2 \pm a^2}) \right]$$

$$165. \int \frac{dx}{\sqrt{(x^2 \pm a^2)^3}} = \frac{\pm x}{a^2 \sqrt{x^2 \pm a^2}}$$

$$166. \int \frac{x dx}{\sqrt{(x^2 \pm a^2)^3}} = \frac{-1}{\sqrt{x^2 \pm a^2}}$$

$$167. \int x\sqrt{(x^2 \pm a^2)^3} dx = \frac{1}{5} \sqrt{(x^2 \pm a^2)^5}$$

$$168. \int x^2 \sqrt{x^2 \pm a^2} dx = \frac{x}{4} \sqrt{(x^2 \pm a^2)^3} \mp \frac{a^2}{8} x \sqrt{x^2 \pm a^2} - \frac{a^4}{8} \log(x + \sqrt{x^2 \pm a^2})$$

$$169. \int x^3 \sqrt{x^2 + a^2} dx = \left(\frac{1}{5}x^2 - \frac{2}{15}a^2\right) \sqrt{(a^2 + x^2)^3}$$

$$170. \int x^3 \sqrt{x^2 - a^2} dx = \frac{1}{5} \sqrt{(x^2 - a^2)^5} + \frac{a^2}{3} \sqrt{(x^2 - a^2)^3}$$

$$171. \int \frac{x^2 dx}{\sqrt{x^2 \pm a^2}} = \frac{x}{2} \sqrt{x^2 \pm a^2} \mp \frac{a^2}{2} \log(x + \sqrt{x^2 \pm a^2})$$

INTEGRALS (Continued)

172.  $\int \frac{x^3 dx}{\sqrt{x^2 \pm a^2}} = \frac{1}{3} \sqrt{(x^2 \pm a^2)^3} \mp a^2 \sqrt{x^2 \pm a^2}$
173.  $\int \frac{dx}{x^2 \sqrt{x^2 \pm a^2}} = \mp \frac{\sqrt{x^2 \pm a^2}}{a^2 x}$
174.  $\int \frac{dx}{x^3 \sqrt{x^2 + a^2}} = \frac{\sqrt{x^2 + a^2}}{2a^2 x^2} + \frac{1}{2a^3} \log \frac{a + \sqrt{x^2 + a^2}}{x}$
175.  $\int \frac{dx}{x^3 \sqrt{x^2 - a^2}} = \frac{\sqrt{x^2 - a^2}}{2a^2 x^2} + \frac{1}{2|a^3|} \sec^{-1} \frac{x}{a}$
176.  $\int x^2 \sqrt{(x^2 \pm a^2)^3} dx = \frac{x}{6} \sqrt{(x^2 \pm a^2)^5} \mp \frac{a^2 x}{24} \sqrt{(x^2 \pm a^2)^3} - \frac{a^4 x}{16} \sqrt{x^2 \pm a^2}$   
 $\mp \frac{a^6}{16} \log(x + \sqrt{x^2 \pm a^2})$
177.  $\int x^3 \sqrt{(x^2 \pm a^2)^3} dx = \frac{1}{7} \sqrt{(x^2 \pm a^2)^7} \mp \frac{a^2}{5} \sqrt{(x^2 \pm a^2)^5}$
178.  $\int \frac{\sqrt{x^2 \pm a^2} dx}{x^2} = -\frac{\sqrt{x^2 \pm a^2}}{x} + \log(x + \sqrt{x^2 \pm a^2})$
179.  $\int \frac{\sqrt{x^2 + a^2}}{x^3} dx = -\frac{\sqrt{x^2 + a^2}}{2x^2} - \frac{1}{2a} \log \frac{a + \sqrt{x^2 + a^2}}{x}$
180.  $\int \frac{\sqrt{x^2 - a^2}}{x^3} dx = -\frac{\sqrt{x^2 - a^2}}{2x^2} + \frac{1}{2|a|} \sec^{-1} \frac{x}{a}$
181.  $\int \frac{\sqrt{x^2 \pm a^2}}{x^4} dx = \mp \frac{\sqrt{(x^2 \pm a^2)^3}}{3a^2 x^3}$
182.  $\int \frac{x^2 dx}{\sqrt{(x^2 \pm a^2)^3}} = \frac{-x}{\sqrt{x^2 \pm a^2}} + \log(x + \sqrt{x^2 \pm a^2})$
183.  $\int \frac{x^3 dx}{\sqrt{(x^2 \pm a^2)^3}} = \sqrt{x^2 \pm a^2} \pm \frac{a^2}{\sqrt{x^2 \pm a^2}}$
184.  $\int \frac{dx}{x \sqrt{(x^2 + a^2)^3}} = \frac{1}{a^2 \sqrt{x^2 + a^2}} - \frac{1}{a^3} \log \frac{a + \sqrt{x^2 + a^2}}{x}$
185.  $\int \frac{dx}{x \sqrt{(x^2 - a^2)^3}} = -\frac{1}{a^2 \sqrt{x^2 - a^2}} - \frac{1}{|a^3|} \sec^{-1} \frac{x}{a}$
186.  $\int \frac{dx}{x^2 \sqrt{(x^2 \pm a^2)^3}} = -\frac{1}{a^4} \left[ \frac{\sqrt{x^2 \pm a^2}}{x} + \frac{x}{\sqrt{x^2 \pm a^2}} \right]$
187.  $\int \frac{dx}{x^3 \sqrt{(x^2 + a^2)^3}} = -\frac{1}{2a^2 x^2 \sqrt{x^2 + a^2}} - \frac{3}{2a^4 \sqrt{x^2 + a^2}} + \frac{3}{2a^5} \log \frac{a + \sqrt{x^2 + a^2}}{x}$
188.  $\int \frac{dx}{x^3 \sqrt{(x^2 - a^2)^3}} = \frac{1}{2a^2 x^2 \sqrt{x^2 - a^2}} - \frac{3}{2a^4 \sqrt{x^2 - a^2}} - \frac{3}{2|a^5|} \sec^{-1} \frac{x}{a}$
189.  $\int \frac{x^m}{\sqrt{x^2 \pm a^2}} dx = \frac{1}{m} x^{m-1} \sqrt{x^2 \pm a^2} \mp \frac{m-1}{m} a^2 \int \frac{x^{m-2}}{\sqrt{x^2 \pm a^2}} dx$

INTEGRALS (Continued)

$$190. \int \frac{x^{2m}}{\sqrt{x^2 \pm a^2}} dx = \frac{(2m)!}{2^{2m}(m!)^2} \left[ \sqrt{x^2 \pm a^2} \sum_{r=1}^m \frac{r!(r-1)!}{(2r)!} (\mp a^2)^{m-r} (2x)^{2r-1} + (\mp a^2)^m \log(x + \sqrt{x^2 \pm a^2}) \right]$$

$$191. \int \frac{x^{2m+1}}{\sqrt{x^2 \pm a^2}} dx = \sqrt{x^2 \pm a^2} \sum_{r=0}^m \frac{(2r)!(m!)^2}{(2m+1)!(r!)^2} (\mp 4a^2)^{m-r} x^{2r}$$

$$192. \int \frac{dx}{x^m \sqrt{x^2 \pm a^2}} = \mp \frac{\sqrt{x^2 \pm a^2}}{(m-1)a^2 x^{m-1}} \mp \frac{(m-2)}{(m-1)a^2} \int \frac{dx}{x^{m-2} \sqrt{x^2 \pm a^2}}$$

$$193. \int \frac{dx}{x^{2m} \sqrt{x^2 \pm a^2}} = \sqrt{x^2 \pm a^2} \sum_{r=0}^{m-1} \frac{(m-1)!m!(2r)!2^{2m-2r-1}}{(r!)^2(2m)!(\mp a^2)^{m-r} x^{2r+1}}$$

$$194. \int \frac{dx}{x^{2m+1} \sqrt{x^2 + a^2}} = \frac{(2m)!}{(m!)^2} \left[ \frac{\sqrt{x^2 + a^2}}{a^2} \sum_{r=1}^m (-1)^{m-r+1} \frac{r!(r-1)!}{2(2r)!(4a^2)^{m-r} x^{2r}} + \frac{(-1)^{m+1}}{2^{2m} a^{2m+1}} \log \frac{\sqrt{x^2 + a^2} + a}{x} \right]$$

$$195. \int \frac{dx}{x^{2m+1} \sqrt{x^2 - a^2}} = \frac{(2m)!}{(m!)^2} \left[ \frac{\sqrt{x^2 - a^2}}{a^2} \sum_{r=1}^m \frac{r!(r-1)!}{2(2r)!(4a^2)^{m-r} x^{2r}} + \frac{1}{2^{2m} |a|^{2m+1}} \sec^{-1} \frac{x}{a} \right]$$

$$196. \int \frac{dx}{(x-a)\sqrt{x^2 - a^2}} = -\frac{\sqrt{x^2 - a^2}}{a(x-a)}$$

$$197. \int \frac{dx}{(x+a)\sqrt{x^2 - a^2}} = \frac{\sqrt{x^2 - a^2}}{a(x+a)}$$

$$198. \int f(x, \sqrt{x^2 + a^2}) dx = a \int f(a \tan u, a \sec u) \sec^2 u du, \quad (u = \tan^{-1} \frac{x}{a}, a > 0)$$

$$199. \int f(x, \sqrt{x^2 - a^2}) dx = a \int f(a \sec u, a \tan u) \sec u \tan u du, \quad (u = \sec^{-1} \frac{x}{a}, a > 0)$$

FORMS CONTAINING  $\sqrt{a^2 - x^2}$

$$200. \int \sqrt{a^2 - x^2} dx = \frac{1}{2} \left[ x\sqrt{a^2 - x^2} + a^2 \sin^{-1} \frac{x}{|a|} \right]$$

$$201. \int \frac{dx}{\sqrt{a^2 - x^2}} = \begin{cases} \sin^{-1} \frac{x}{|a|} \\ \text{or} \\ -\cos^{-1} \frac{x}{|a|} \end{cases}$$

$$202. \int \frac{dx}{x\sqrt{a^2 - x^2}} = -\frac{1}{a} \log \left( \frac{a + \sqrt{a^2 - x^2}}{x} \right)$$

$$203. \int \frac{\sqrt{a^2 - x^2}}{x} dx = \sqrt{a^2 - x^2} - a \log \left( \frac{a + \sqrt{a^2 - x^2}}{x} \right)$$

$$204. \int \frac{x dx}{\sqrt{a^2 - x^2}} = -\sqrt{a^2 - x^2}$$

$$205. \int x\sqrt{a^2 - x^2} dx = -\frac{1}{3} \sqrt{(a^2 - x^2)^3}$$

INTEGRALS (Continued)

206.  $\int \sqrt{(a^2 - x^2)^3} dx = \frac{1}{4} \left[ x\sqrt{(a^2 - x^2)^3} + \frac{3a^2x}{2}\sqrt{a^2 - x^2} + \frac{3a^4}{2} \sin^{-1} \frac{x}{|a|} \right]$
207.  $\int \frac{dx}{\sqrt{(a^2 - x^2)^3}} = \frac{x}{a^2\sqrt{a^2 - x^2}}$
208.  $\int \frac{x dx}{\sqrt{(a^2 - x^2)^3}} = \frac{1}{\sqrt{a^2 - x^2}}$
209.  $\int x\sqrt{(a^2 - x^2)^3} dx = -\frac{1}{5}\sqrt{(a^2 - x^2)^5}$
210.  $\int x^2\sqrt{a^2 - x^2} dx = -\frac{x}{4}\sqrt{(a^2 - x^2)^3} + \frac{a^2}{8} \left( x\sqrt{a^2 - x^2} + a^2 \sin^{-1} \frac{x}{|a|} \right)$
211.  $\int x^3\sqrt{a^2 - x^2} dx = \left(-\frac{1}{5}x^2 - \frac{2}{15}a^2\right)\sqrt{(a^2 - x^2)^3}$
212.  $\int x^2\sqrt{(a^2 - x^2)^3} dx = -\frac{1}{6}x\sqrt{(a^2 - x^2)^5} + \frac{a^2x}{24}\sqrt{(a^2 - x^2)^3} + \frac{a^4x}{16}\sqrt{a^2 - x^2} + \frac{a^6}{16} \sin^{-1} \frac{x}{|a|}$
213.  $\int x^3\sqrt{(a^2 - x^2)^3} dx = \frac{1}{7}\sqrt{(a^2 - x^2)^7} - \frac{a^2}{5}\sqrt{(a^2 - x^2)^5}$
214.  $\int \frac{x^2 dx}{\sqrt{a^2 - x^2}} = -\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{|a|}$
215.  $\int \frac{dx}{x^2\sqrt{a^2 - x^2}} = -\frac{\sqrt{a^2 - x^2}}{a^2x}$
216.  $\int \frac{\sqrt{a^2 - x^2}}{x^2} dx = -\frac{\sqrt{a^2 - x^2}}{x} - \sin^{-1} \frac{x}{|a|}$
217.  $\int \frac{\sqrt{a^2 - x^2}}{x^3} dx = -\frac{\sqrt{a^2 - x^2}}{2x^2} + \frac{1}{2a} \log \frac{a + \sqrt{a^2 - x^2}}{x}$
218.  $\int \frac{\sqrt{a^2 - x^2}}{x^4} dx = -\frac{\sqrt{(a^2 - x^2)^3}}{3a^2x^3}$
219.  $\int \frac{x^2 dx}{\sqrt{(a^2 - x^2)^3}} = \frac{x}{\sqrt{a^2 - x^2}} - \sin^{-1} \frac{x}{|a|}$
220.  $\int \frac{x^3 dx}{\sqrt{a^2 - x^2}} = -\frac{2}{3}(a^2 - x^2)^{3/2} - x^2(a^2 - x^2)^{1/2} = -\frac{1}{3}\sqrt{a^2 - x^2}(x^2 + 2a^2)$
221.  $\int \frac{x^3 dx}{\sqrt{(a^2 - x^2)^3}} = 2(a^2 - x^2)^{1/2} + \frac{x^2}{(a^2 - x^2)^{1/2}} = -\frac{a^2}{\sqrt{a^2 - x^2}} + \sqrt{a^2 - x^2}$
222.  $\int \frac{dx}{x^3\sqrt{a^2 - x^2}} = -\frac{\sqrt{a^2 - x^2}}{2a^2x^2} - \frac{1}{2a^3} \log \frac{a + \sqrt{a^2 - x^2}}{x}$
223.  $\int \frac{dx}{x\sqrt{(a^2 - x^2)^3}} = \frac{1}{a^2\sqrt{a^2 - x^2}} - \frac{1}{a^3} \log \frac{a + \sqrt{a^2 - x^2}}{x}$
224.  $\int \frac{dx}{x^2\sqrt{(a^2 - x^2)^3}} = \frac{1}{a^4} \left[ -\frac{\sqrt{a^2 - x^2}}{x} + \frac{x}{\sqrt{a^2 - x^2}} \right]$
225.  $\int \frac{dx}{x^3\sqrt{(a^2 - x^2)^3}} = -\frac{1}{2a^2x^2\sqrt{a^2 - x^2}} + \frac{3}{2a^4\sqrt{a^2 - x^2}} - \frac{3}{2a^5} \log \frac{a + \sqrt{a^2 - x^2}}{x}$

INTEGRALS (Continued)

226.  $\int \frac{x^m}{\sqrt{a^2 - x^2}} dx = -\frac{x^{m-1}\sqrt{a^2 - x^2}}{m} + \frac{(m-1)a^2}{m} \int \frac{x^{m-2}}{\sqrt{a^2 - x^2}} dx$
227.  $\int \frac{x^{2m}}{\sqrt{a^2 - x^2}} dx = \frac{(2m)!}{(m!)^2} \left[ -\sqrt{a^2 - x^2} \sum_{r=1}^m \frac{r!(r-1)!}{2^{2m-2r+1}(2r)!} a^{2m-2r} x^{2r-1} + \frac{a^{2m}}{2^{2m}} \sin^{-1} \frac{x}{|a|} \right]$
228.  $\int \frac{x^{2m+1}}{\sqrt{a^2 - x^2}} dx = -\sqrt{a^2 - x^2} \sum_{r=0}^m \frac{(2r)!(m!)^2}{(2m+1)!(r!)^2} (4a^2)^{m-r} x^{2r}$
229.  $\int \frac{dx}{x^m \sqrt{a^2 - x^2}} = -\frac{\sqrt{a^2 - x^2}}{(m-1)a^2 x^{m-1}} + \frac{m-2}{(m-1)a^2} \int \frac{dx}{x^{m-2} \sqrt{a^2 - x^2}}$
230.  $\int \frac{ax}{x^{2m} \sqrt{a^2 - x^2}} = -\sqrt{a^2 - x^2} \sum_{r=0}^{m-1} \frac{(m-1)!m!(2r)!2^{2m-2r-1}}{(r!)^2(2m)!a^{2m-2r}x^{2r+1}}$
231.  $\int \frac{dx}{x^{2m+1} \sqrt{a^2 - x^2}} = \frac{(2m)!}{(m!)^2} \left[ -\frac{\sqrt{a^2 - x^2}}{a^2} \sum_{r=1}^m \frac{r!(r-1)!}{2(2r)!(4a^2)^{m-r}x^{2r}} + \frac{1}{2^{2m}a^{2m+1}} \log \frac{a - \sqrt{a^2 - x^2}}{x} \right]$
232.  $\int \frac{dx}{(b^2 - x^2)\sqrt{a^2 - x^2}} = \frac{1}{2b\sqrt{a^2 - b^2}} \log \frac{(b\sqrt{a^2 - x^2} + x\sqrt{a^2 - b^2})^2}{b^2 - x^2}, \quad (a^2 > b^2)$
233.  $\int \frac{dx}{(b^2 - x^2)\sqrt{a^2 - x^2}} = \frac{1}{b\sqrt{b^2 - a^2}} \tan^{-1} \frac{x\sqrt{b^2 - a^2}}{b\sqrt{a^2 - x^2}}, \quad (b^2 > a^2)$
234.  $\int \frac{dx}{(b^2 + x^2)\sqrt{a^2 - x^2}} = \frac{1}{b\sqrt{a^2 + b^2}} \tan^{-1} \frac{x\sqrt{a^2 + b^2}}{b\sqrt{a^2 - x^2}}$
235.  $\int \frac{\sqrt{a^2 - x^2}}{b^2 + x^2} dx = \frac{\sqrt{a^2 + b^2}}{|b|} \sin^{-1} \frac{x\sqrt{a^2 + b^2}}{|a|\sqrt{x^2 + b^2}} - \sin^{-1} \frac{x}{|a|}$
236.  $\int f(x, \sqrt{a^2 - x^2}) dx = a \int f(a \sin u, a \cos u) \cos u du, \quad (u = \sin^{-1} \frac{x}{a}, a > 0)$

FORMS CONTAINING  $\sqrt{a + bx + cx^2}$

$$X = a + bx + cx^2, \quad q = 4ac - b^2, \quad \text{and } k = \frac{4c}{q}$$

If  $q = 0$ , then  $\sqrt{X} = \sqrt{c} \left| x + \frac{b}{2c} \right|$

$$237. \int \frac{dx}{\sqrt{X}} = \begin{cases} \frac{1}{\sqrt{c}} \log(2\sqrt{cX} + 2cx + b) \\ \text{or} \\ \frac{1}{\sqrt{c}} \sinh^{-1} \frac{2cx + b}{\sqrt{q}}, \quad (c > 0) \end{cases}$$

$$238. \int \frac{dx}{\sqrt{X}} = -\frac{1}{\sqrt{-c}} \sin^{-1} \frac{2cx + b}{\sqrt{-q}}, \quad (c < 0)$$

$$239. \int \frac{dx}{X\sqrt{X}} = \frac{2(2cx + b)}{q\sqrt{X}}$$

INTEGRALS (Continued)

$$240. \int \frac{dx}{X^2\sqrt{X}} = \frac{2(2cx+b)}{3q\sqrt{X}} \left( \frac{1}{X} + 2k \right)$$

$$241. \int \frac{dx}{X^n\sqrt{X}} = \begin{cases} \frac{2(2cx+b)\sqrt{X}}{(2n-1)qX^n} + \frac{2k(n-1)}{2n-1} \int \frac{dx}{X^{n-1}\sqrt{X}} \\ \text{or} \\ \frac{(2cx+b)(n!)(n-1)!4^n k^{n-1}}{q[(2n)!]\sqrt{X}} \sum_{r=0}^{n-1} \frac{(2r)!}{(4kX)^r (r!)^2} \end{cases}$$

$$242. \int \sqrt{X} dx = \frac{(2cx+b)\sqrt{X}}{4c} + \frac{1}{2k} \int \frac{dx}{\sqrt{X}}$$

$$243. \int X\sqrt{X} dx = \frac{(2cx+b)\sqrt{X}}{8c} \left( X + \frac{3}{2k} \right) + \frac{3}{8k^2} \int \frac{dx}{\sqrt{X}}$$

$$244. \int X^2\sqrt{X} dx = \frac{(2cx+b)\sqrt{X}}{12c} \left( X^2 + \frac{5X}{4k} + \frac{15}{8k^2} \right) + \frac{5}{16k^3} \int \frac{dx}{\sqrt{X}}$$

$$245. \int X^n\sqrt{X} dx = \begin{cases} \frac{(2cx+b)X^n\sqrt{X}}{4(n+1)c} + \frac{2n+1}{2(n+1)k} \int X^{n-1}\sqrt{X} dx \\ \text{or} \\ \frac{(2n+2)!}{[(n+1)!]^2(4k)^{n+1}} \left[ \frac{k(2cx+b)\sqrt{X}}{c} \sum_{r=0}^n \frac{r!(r+1)!(4kX)^r}{(2r+2)!} + \int \frac{dx}{\sqrt{X}} \right] \end{cases}$$

$$246. \int \frac{x dx}{\sqrt{X}} = \frac{\sqrt{X}}{c} - \frac{b}{2c} \int \frac{dx}{\sqrt{X}}$$

$$247. \int \frac{x dx}{X\sqrt{X}} = -\frac{2(bx+2a)}{q\sqrt{X}}$$

$$248. \int \frac{x dx}{X^n\sqrt{X}} = -\frac{\sqrt{X}}{(2n-1)cX^n} - \frac{b}{2c} \int \frac{dx}{X^n\sqrt{X}}$$

$$249. \int \frac{x^2 dx}{\sqrt{X}} = \left( \frac{x}{2c} - \frac{3b}{4c^2} \right) \sqrt{X} + \frac{3b^2 - 4ac}{8c^2} \int \frac{dx}{\sqrt{X}}$$

$$250. \int \frac{x^2 dx}{X\sqrt{X}} = \frac{(2b^2 - 4ac)x + 2ab}{cq\sqrt{X}} + \frac{1}{c} \int \frac{dx}{\sqrt{X}}$$

$$251. \int \frac{x^2 dx}{X^n\sqrt{X}} = \frac{(2b^2 - 4ac)x + 2ab}{(2n-1)cq X^{n-1}\sqrt{X}} + \frac{4ac + (2n-3)b^2}{(2n-1)cq} \int \frac{dx}{X^{n-1}\sqrt{X}}$$

$$252. \int \frac{x^3 dx}{\sqrt{X}} = \left( \frac{x^2}{3c} - \frac{5bx}{12c^2} + \frac{5b^2}{8c^3} - \frac{2a}{3c^2} \right) \sqrt{X} + \left( \frac{3ab}{4c^2} - \frac{5b^3}{16c^3} \right) \int \frac{dx}{\sqrt{X}}$$

$$253. \int \frac{x^n dx}{\sqrt{X}} = \frac{1}{nc} x^{n-1} \sqrt{X} - \frac{(2n-1)b}{2nc} \int \frac{x^{n-1} dx}{\sqrt{X}} - \frac{(n-1)a}{nc} \int \frac{x^{n-2} dx}{\sqrt{X}}$$

$$254. \int x\sqrt{X} dx = \frac{X\sqrt{X}}{3c} - \frac{b(2cx+b)}{8c^2} \sqrt{X} - \frac{b}{4ck} \int \frac{dx}{\sqrt{X}}$$

$$255. \int xX\sqrt{X} dx = \frac{X^2\sqrt{X}}{5c} - \frac{b}{2c} \int X\sqrt{X} dx$$



INTEGRALS (Continued)

256.  $\int xX^n\sqrt{X} dx = \frac{X^{n+1}\sqrt{X}}{(2n+3)c} - \frac{b}{2c} \int X^n\sqrt{X} dx$
257.  $\int x^2\sqrt{X} dx = \left(x - \frac{5b}{6c}\right) \frac{X\sqrt{X}}{4c} + \frac{5b^2 - 4ac}{16c^2} \int \sqrt{X} dx$
258.  $\int \frac{dx}{x\sqrt{X}} = -\frac{1}{\sqrt{a}} \log \frac{2\sqrt{aX} + bx + 2a}{x}, \quad (a > 0)$
259.  $\int \frac{dx}{x\sqrt{X}} = \frac{1}{\sqrt{-a}} \sin^{-1} \left( \frac{bx + 2a}{|x|\sqrt{-a}} \right), \quad (a < 0)$
260.  $\int \frac{dx}{x\sqrt{X}} = -\frac{2\sqrt{X}}{bx}, \quad (a = 0)$
261.  $\int \frac{dx}{x^2\sqrt{X}} = -\frac{\sqrt{X}}{ax} - \frac{b}{2a} \int \frac{dx}{x\sqrt{X}}$
262.  $\int \frac{\sqrt{X} dx}{x} = \sqrt{X} + \frac{b}{2} \int \frac{dx}{\sqrt{X}} + a \int \frac{dx}{x\sqrt{X}}$
263.  $\int \frac{\sqrt{X} dx}{x^2} = -\frac{\sqrt{X}}{x} + \frac{b}{2} \int \frac{dx}{x\sqrt{X}} + c \int \frac{dx}{\sqrt{X}}$

FORMS INVOLVING  $\sqrt{2ax - x^2}$

264.  $\int \sqrt{2ax - x^2} dx = \frac{1}{2} \left[ (x - a)\sqrt{2ax - x^2} + a^2 \sin^{-1} \frac{x - a}{|a|} \right]$
265.  $\int \frac{dx}{\sqrt{2ax - x^2}} = \begin{cases} \cos^{-1} \frac{a - x}{|a|} \\ \text{or} \\ \sin^{-1} \frac{x - a}{|a|} \end{cases}$
266.  $\int x^n \sqrt{2ax - x^2} dx = \begin{cases} -\frac{x^{n-1}(2ax - x^2)^{3/2}}{n+2} + \frac{(2n+1)a}{n+2} \int x^{n-1} \sqrt{2ax - x^2} dx \\ \text{or} \\ \sqrt{2ax - x^2} \left[ \frac{x^{n+1}}{n+2} - \sum_{r=0}^n \frac{(2n+1)!(r!)^2 a^{n-r+1}}{2^{n-r}(2r)!(n+2)!n!} x^r \right] \\ + \frac{(2n+1)!a^{n+2}}{2^n n!(n+2)!} \sin^{-1} \frac{x - a}{|a|} \end{cases}$
267.  $\int \frac{\sqrt{2ax - x^2}}{x^n} dx = \frac{(2ax - x^2)^{3/2}}{(3-2n)ax^n} + \frac{n-3}{(2n-3)a} \int \frac{\sqrt{2ax - x^2}}{x^{n-1}} dx$
268.  $\int \frac{x^n dx}{\sqrt{2ax - x^2}} = \begin{cases} \frac{-x^{n-1}\sqrt{2ax - x^2}}{n} + \frac{a(2n-1)}{n} \int \frac{x^{n-1}}{\sqrt{2ax - x^2}} dx \\ \text{or} \\ -\sqrt{2ax - x^2} \sum_{r=1}^n \frac{(2n)!(r-1)!a^{n-r}}{2^{n-r}(2r)!(n!)^2} x^{r-1} + \frac{(2n)!a^n}{2^n(n!)^2} \sin^{-1} \frac{x - a}{|a|} \end{cases}$

INTEGRALS (Continued)

$$269. \int \frac{dx}{x^n \sqrt{2ax - x^2}} = \begin{cases} \frac{\sqrt{2ax - x^2}}{a(1 - 2n)x^n} + \frac{n - 1}{(2n - 1)a} \int \frac{dx}{x^{n-1} \sqrt{2ax - x^2}} \\ \text{or} \\ -\sqrt{2ax - x^2} \sum_{r=0}^{n-1} \frac{2^{n-r}(n-1)!n!(2r)!}{(2n)!(r!)^2 a^{n-r} x^{r+1}} \end{cases}$$

$$270. \int \frac{dx}{(2ax - x^2)^{3/2}} = \frac{x - a}{a^2 \sqrt{2ax - x^2}}$$

$$271. \int \frac{x dx}{(2ax - x^2)^{3/2}} = \frac{x}{a \sqrt{2ax - x^2}}$$

MISCELLANEOUS ALGEBRAIC FORMS

$$272. \int \frac{dx}{\sqrt{2ax + x^2}} = \log(x + a + \sqrt{2ax + x^2})$$

$$273. \int \sqrt{ax^2 + c} dx = \frac{x}{2} \sqrt{ax^2 + c} + \frac{c}{2\sqrt{a}} \log(x\sqrt{a} + \sqrt{ax^2 + c}), \quad (a > 0)$$

$$274. \int \sqrt{ax^2 + c} dx = \frac{x}{2} \sqrt{ax^2 + c} + \frac{c}{2\sqrt{-a}} \sin^{-1}\left(x\sqrt{-\frac{a}{c}}\right), \quad (a < 0)$$

$$275. \int \sqrt{\frac{1+x}{1-x}} dx = \sin^{-1} x - \sqrt{1-x^2}$$

$$276. \int \frac{dx}{x\sqrt{ax^n + c}} = \begin{cases} \frac{1}{n\sqrt{c}} \log \frac{\sqrt{ax^n + c} - \sqrt{c}}{\sqrt{ax^n + c} + \sqrt{c}} \\ \text{or} \\ \frac{2}{n\sqrt{c}} \log \frac{\sqrt{ax^n + c} - \sqrt{c}}{\sqrt{x^n}}, \quad (c > 0) \end{cases}$$

$$277. \int \frac{dx}{x\sqrt{ax^n + c}} = \frac{2}{n\sqrt{-c}} \sec^{-1} \sqrt{-\frac{ax^n}{c}}, \quad (c < 0)$$

$$278. \int \frac{dx}{\sqrt{ax^2 + c}} = \frac{1}{\sqrt{a}} \log(x\sqrt{a} + \sqrt{ax^2 + c}), \quad (a > 0)$$

$$279. \int \frac{dx}{\sqrt{ax^2 + c}} = \frac{1}{\sqrt{-a}} \sin^{-1}\left(x\sqrt{-\frac{a}{c}}\right), \quad (a < 0)$$

$$280. \int (ax^2 + c)^{m+1/2} dx = \begin{cases} \frac{x(ax^2 + c)^{m+1/2}}{2(m+1)} + \frac{(2m+1)c}{2(m+1)} \int (ax^2 + c)^{m-1/2} dx \\ \text{or} \\ x\sqrt{ax^2 + c} \sum_{r=0}^m \frac{(2m+1)!(r!)^2 c^{m-r}}{2^{2m-2r+1} m!(m+1)!(2r+1)!} (ax^2 + c)^r \\ + \frac{(2m+1)!c^{m+1}}{2^{2m+1} m!(m+1)!} \int \frac{dx}{\sqrt{ax^2 + c}} \end{cases}$$

$$281. \int x(ax^2 + c)^{m+1/2} dx = \frac{(ax^2 + c)^{m+3/2}}{(2m+3)a}$$

INTEGRALS (Continued)

$$282. \int \frac{(ax^2 + c)^{m+1/2}}{x} dx = \begin{cases} \frac{(ax^2 + c)^{m+1/2}}{2m+1} + c \int \frac{(ax^2 + c)^{m-1/2}}{x} dx \\ \text{or} \\ \sqrt{ax^2 + c} \sum_{r=0}^m \frac{c^{m-r}(ax^2 + c)^r}{2r+1} + c^{m+1} \int \frac{dx}{x\sqrt{ax^2 + c}} \end{cases}$$

$$283. \int \frac{dx}{(ax^2 + c)^{m+1/2}} = \begin{cases} \frac{x}{(2m-1)c(ax^2 + c)^{m-1/2}} + \frac{2m-2}{(2m-1)c} \int \frac{dx}{(ax^2 + c)^{m-1/2}} \\ \text{or} \\ \frac{x}{\sqrt{ax^2 + c}} \sum_{r=0}^{m-1} \frac{2^{2m-2r-1}(m-1)!m!(2r)!}{(2m)!(r!)^2 c^{m-r}(ax^2 + c)^r} \end{cases}$$

$$284. \int \frac{dx}{x^m \sqrt{ax^2 + c}} = -\frac{\sqrt{ax^2 + c}}{(m-1)cx^{m-1}} - \frac{(m-2)a}{(m-1)c} \int \frac{dx}{x^{m-2} \sqrt{ax^2 + c}}$$

$$285. \int \frac{1+x^2}{(1-x^2)\sqrt{1+x^4}} dx = \frac{1}{\sqrt{2}} \log \frac{x\sqrt{2} + \sqrt{1+x^4}}{1-x^2}$$

$$286. \int \frac{1-x^2}{(1+x^2)\sqrt{1+x^4}} dx = \frac{1}{\sqrt{2}} \tan^{-1} \frac{x\sqrt{2}}{\sqrt{1+x^4}}$$

$$287. \int \frac{dx}{x\sqrt{x^n + a^2}} = -\frac{2}{na} \log \frac{a + \sqrt{x^n + a^2}}{\sqrt{x^n}}$$

$$288. \int \frac{dx}{x\sqrt{x^n - a^2}} = -\frac{2}{na} \sin^{-1} \frac{a}{\sqrt{x^n}}$$

$$289. \int \sqrt{\frac{x}{a^3 - x^3}} dx = \frac{2}{3} \sin^{-1} \left( \frac{x}{a} \right)^{3/2}$$

FORMS INVOLVING TRIGONOMETRIC FUNCTIONS

$$290. \int (\sin ax) dx = -\frac{1}{a} \cos ax$$

$$291. \int (\cos ax) dx = \frac{1}{a} \sin ax$$

$$292. \int (\tan ax) dx = -\frac{1}{a} \log \cos ax = \frac{1}{a} \log \sec ax$$

$$293. \int (\cot ax) dx = \frac{1}{a} \log \sin ax = -\frac{1}{a} \log \csc ax$$

$$294. \int (\sec ax) dx = \frac{1}{a} \log(\sec ax + \tan ax) = \frac{1}{a} \log \tan \left( \frac{\pi}{4} + \frac{ax}{2} \right)$$

$$295. \int (\csc ax) dx = \frac{1}{a} \log(\csc ax - \cot ax) = \frac{1}{a} \log \tan \frac{ax}{2}$$

$$296. \int (\sin^2 ax) dx = -\frac{1}{2a} \cos ax \sin ax + \frac{1}{2} x = \frac{1}{2} x - \frac{1}{4a} \sin 2ax$$

$$297. \int (\sin^3 ax) dx = -\frac{1}{3a} (\cos ax)(\sin^2 ax + 2)$$

$$298. \int (\sin^4 ax) dx = \frac{3x}{8} - \frac{\sin 2ax}{4a} + \frac{\sin 4ax}{32a}$$

$$299. \int (\sin^n ax) dx = -\frac{\sin^{n-1} ax \cos ax}{na} + \frac{n-1}{n} \int (\sin^{n-2} ax) dx$$

INTEGRALS (Continued)

300.  $\int (\sin^{2m} ax) dx = -\frac{\cos ax}{a} \sum_{r=0}^{m-1} \frac{(2m)!(r!)^2}{2^{2m-2r}(2r+1)(m!)^2} \sin^{2r+1} ax + \frac{(2m)!}{2^{2m}(m!)^2} x$
301.  $\int (\sin^{2m+1} ax) dx = -\frac{\cos ax}{a} \sum_{r=0}^m \frac{2^{2m-2r}(m!)^2(2r)!}{(2m+1)(r!)^2} \sin^{2r} ax$
302.  $\int (\cos^2 ax) dx = \frac{1}{2a} \sin ax \cos ax + \frac{1}{2} x = \frac{1}{2} x + \frac{1}{4a} \sin 2ax$
303.  $\int (\cos^3 ax) dx = \frac{1}{3a} (\sin ax)(\cos^2 ax + 2)$
304.  $\int (\cos^4 ax) dx = \frac{3x}{8} + \frac{\sin 2ax}{4a} + \frac{\sin 4ax}{32a}$
305.  $\int (\cos^n ax) dx = \frac{1}{na} \cos^{n-1} ax \sin ax + \frac{n-1}{n} \int (\cos^{n-2} ax) dx$
306.  $\int (\cos^{2m} ax) dx = \frac{\sin ax}{a} \sum_{r=0}^{m-1} \frac{(2m)!(r!)^2}{2^{2m-2r}(2r+1)(m!)^2} \cos^{2r+1} ax + \frac{(2m)!}{2^{2m}(m!)^2} x$
307.  $\int (\cos^{2m+1} ax) dx = \frac{\sin ax}{a} \sum_{r=0}^m \frac{2^{2m-2r}(m!)^2(2r)!}{(2m+1)(r!)^2} \cos^{2r} ax$
308.  $\int \frac{dx}{\sin^2 ax} = \int (\csc^2 ax) dx = -\frac{1}{a} \cot ax$
309.  $\int \frac{dx}{\sin^m ax} = \int (\csc^m ax) dx = -\frac{1}{(m-1)a} \cdot \frac{\cos ax}{\sin^{m-1} ax} + \frac{m-2}{m-1} \int \frac{dx}{\sin^{m-2} ax}$
310.  $\int \frac{dx}{\sin^{2m} ax} = \int (\csc^{2m} ax) dx = -\frac{1}{a} \cos ax \sum_{r=0}^{m-1} \frac{2^{2m-2r-1}(m-1)!m!(2r)!}{(2m)!(r!)^2 \sin^{2r+1} ax}$
311.  $\int \frac{dx}{\sin^{2m+1} ax} = \int (\csc^{2m+1} ax) dx$   
 $= -\frac{1}{a} \cos ax \sum_{r=0}^{m-1} \frac{(2m)!(r!)^2}{2^{2m-2r}(m!)^2(2r+1)! \sin^{2r+2} ax} + \frac{1}{a} \cdot \frac{(2m)!}{2^{2m}(m!)^2} \log \tan \frac{ax}{2}$
312.  $\int \frac{dx}{\cos^2 ax} = \int (\sec^2 ax) dx = \frac{1}{a} \tan ax$
313.  $\int \frac{dx}{\cos^n ax} = \int (\sec^n ax) dx = \frac{1}{(n-1)a} \cdot \frac{\sin ax}{\cos^{n-1} ax} + \frac{n-2}{n-1} \int \frac{dx}{\cos^{n-2} ax}$
314.  $\int \frac{dx}{\cos^{2m} ax} = \int (\sec^{2m} ax) dx = \frac{1}{a} \sin ax \sum_{r=0}^{m-1} \frac{2^{2m-2r-1}(m-1)!m!(2r)!}{(2m)!(r!)^2 \cos^{2r+1} ax}$
315.  $\int \frac{dx}{\cos^{2m+1} ax} = \int (\sec^{2m+1} ax) dx$   
 $= \frac{1}{a} \sin ax \sum_{r=0}^{m-1} \frac{(2m)!(r!)^2}{2^{2m-2r}(m!)^2(2r+1)! \cos^{2r+2} ax} + \frac{1}{a} \cdot \frac{(2m)!}{2^{2m}(m!)^2} \log(\sec ax + \tan ax)$
316.  $\int (\sin mx)(\sin nx) dx = \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)}, \quad (m^2 \neq n^2)$
317.  $\int (\cos mx)(\cos nx) dx = \frac{\sin(m-n)x}{2(m-n)} + \frac{\sin(m+n)x}{2(m+n)}, \quad (m^2 \neq n^2)$
318.  $\int (\sin ax)(\cos ax) dx = \frac{1}{2a} \sin^2 ax$

INTEGRALS (Continued)

$$319. \int (\sin mx)(\cos nx) dx = -\frac{\cos(m-n)x}{2(m-n)} - \frac{\cos(m+n)x}{2(m+n)}, \quad (m^2 \neq n^2)$$

$$320. \int (\sin^2 ax)(\cos^2 ax) dx = -\frac{1}{32a} \sin 4ax + \frac{x}{8}$$

$$321. \int (\sin ax)(\cos^m ax) dx = -\frac{\cos^{m+1} ax}{(m+1)a}$$

$$322. \int (\sin^m ax)(\cos ax) dx = \frac{\sin^{m+1} ax}{(m+1)a}$$

$$323. \int (\cos^m ax)(\sin^n ax) dx = \begin{cases} \frac{\cos^{m-1} ax \sin^{n+1} ax}{(m+n)a} + \frac{m-1}{m+n} \int (\cos^{m-2} ax)(\sin^n ax) dx \\ \text{or} \\ -\frac{\sin^{n-1} ax \cos^{m+1} ax}{(m+n)a} + \frac{n-1}{m+n} \int (\cos^m ax)(\sin^{n-2} ax) dx \end{cases}$$

$$324. \int \frac{\cos^m ax}{\sin^n ax} dx = \begin{cases} -\frac{\cos^{m+1} ax}{(n-1)a \sin^{n-1} ax} - \frac{m-n+2}{n-1} \int \frac{\cos^m ax}{\sin^{n-2} ax} dx \\ \text{or} \\ \frac{\cos^{m-1} ax}{a(m-n) \sin^{n-1} ax} + \frac{m-1}{m-n} \int \frac{\cos^{m-2} ax}{\sin^n ax} dx \end{cases}$$

$$325. \int \frac{\sin^m ax}{\cos^n ax} dx = \begin{cases} \frac{\sin^{m+1} ax}{a(n-1) \cos^{n-1} ax} - \frac{m-n+2}{n-1} \int \frac{\sin^m ax}{\cos^{n-2} ax} dx \\ \text{or} \\ -\frac{\sin^{m-1} ax}{a(m-n) \cos^{n-1} ax} + \frac{m-1}{m-n} \int \frac{\sin^{m-2} ax}{\cos^n ax} dx \end{cases}$$

$$326. \int \frac{\sin ax}{\cos^2 ax} dx = \frac{1}{a \cos ax} = \frac{\sec ax}{a}$$

$$327. \int \frac{\sin^2 ax}{\cos ax} dx = -\frac{1}{a} \sin ax + \frac{1}{a} \log \tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)$$

$$328. \int \frac{\cos ax}{\sin^2 ax} dx = -\frac{1}{a \sin ax} = -\frac{\csc ax}{a}$$

$$329. \int \frac{dx}{(\sin ax)(\cos ax)} = \frac{1}{a} \log \tan ax$$

$$330. \int \frac{dx}{(\sin ax)(\cos^2 ax)} = \frac{1}{a} \left( \sec ax + \log \tan \frac{ax}{2} \right)$$

$$331. \int \frac{dx}{(\sin ax)(\cos^n ax)} = \frac{1}{a(n-1) \cos^{n-1} ax} + \int \frac{dx}{(\sin ax)(\cos^{n-2} ax)}$$

$$332. \int \frac{dx}{(\sin^2 ax)(\cos ax)} = -\frac{1}{a} \csc ax + \frac{1}{a} \log \tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)$$

$$333. \int \frac{dx}{(\sin^2 ax)(\cos^2 ax)} = -\frac{2}{a} \cot 2ax$$

INTEGRALS (Continued)

$$334. \int \frac{dx}{\sin^m ax \cos^n ax} = \begin{cases} -\frac{1}{a(m-1)(\sin^{m-1} ax)(\cos^{n-1} ax)} \\ \quad + \frac{m+n-2}{m-1} \int \frac{dx}{(\sin^{m-2} ax)(\cos^n ax)} \\ \text{or} \\ \frac{1}{a(n-1)\sin^{m-1} ax \cos^{n-1} ax} - \frac{m+n-2}{n-1} \int \frac{dx}{\sin^m ax \cos^{n-2} ax} \end{cases}$$

$$335. \int \sin(a+bx) dx = -\frac{1}{b} \cos(a+bx)$$

$$336. \int \cos(a+bx) dx = \frac{1}{b} \sin(a+bx)$$

$$337. \int \frac{dx}{1 \pm \sin ax} = \mp \frac{1}{a} \tan\left(\frac{\pi}{4} \mp \frac{ax}{2}\right)$$

$$338. \int \frac{dx}{1 + \cos ax} = \frac{1}{a} \tan \frac{ax}{2}$$

$$339. \int \frac{dx}{1 - \cos ax} = -\frac{1}{a} \cot \frac{ax}{2}$$

$$*340. \int \frac{dx}{a+b \sin x} = \begin{cases} \frac{2}{\sqrt{a^2-b^2}} \tan^{-1} \frac{a \tan \frac{x}{2} + b}{\sqrt{a^2-b^2}} \\ \text{or} \\ \frac{1}{\sqrt{b^2-a^2}} \log \frac{a \tan \frac{x}{2} + b - \sqrt{b^2-a^2}}{a \tan \frac{x}{2} + b + \sqrt{b^2-a^2}} \end{cases}$$

$$*341. \int \frac{dx}{a+b \cos x} = \begin{cases} \frac{2}{\sqrt{a^2-b^2}} \tan^{-1} \frac{\sqrt{a^2-b^2} \tan \frac{x}{2}}{a+b} \\ \text{or} \\ \frac{1}{\sqrt{b^2-a^2}} \log \left( \frac{\sqrt{b^2-a^2} \tan \frac{x}{2} + a + b}{\sqrt{b^2-a^2} \tan \frac{x}{2} - a - b} \right) \end{cases}$$

$$*342. \int \frac{dx}{a+b \sin x + c \cos x} = \begin{cases} \frac{1}{\sqrt{b^2+c^2-a^2}} \log \frac{b - \sqrt{b^2+c^2-a^2} + (a-c) \tan \frac{x}{2}}{b + \sqrt{b^2+c^2-a^2} + (a-c) \tan \frac{x}{2}}, & \text{if } a^2 < b^2 + c^2, a \neq c \\ \text{or} \\ \frac{2}{\sqrt{a^2-b^2-c^2}} \tan^{-1} \frac{b + (a-c) \tan \frac{x}{2}}{\sqrt{a^2-b^2-c^2}}, & \text{if } a^2 > b^2 + c^2 \\ \text{or} \\ \frac{1}{a} \left[ \frac{a - (b+c) \cos x - (b-c) \sin x}{a - (b-c) \cos x + (b+c) \sin x} \right], & \text{if } a^2 = b^2 + c^2, a \neq c. \end{cases}$$

\*See note 6 on page A-19.

INTEGRALS (Continued)

$$*343. \int \frac{\sin^2 x \, dx}{a + b \cos^2 x} = \frac{1}{b} \sqrt{\frac{a+b}{a}} \tan^{-1} \left( \sqrt{\frac{a}{a+b}} \tan x \right) - \frac{x}{b}, \quad (ab > 0, \text{ or } |a| > |b|)$$

$$*344. \int \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{1}{ab} \tan^{-1} \left( \frac{b \tan x}{a} \right)$$

$$*345. \int \frac{\cos^2 cx}{a^2 + b^2 \sin^2 cx} dx = \frac{\sqrt{a^2 + b^2}}{ab^2 c} \tan^{-1} \frac{\sqrt{a^2 + b^2} \tan cx}{a} - \frac{x}{b^2}$$

$$346. \int \frac{\sin cx \cos cx}{a \cos^2 cx + b \sin^2 cx} dx = \frac{1}{2c(b-a)} \log(a \cos^2 cx + b \sin^2 cx)$$

$$347. \int \frac{\cos cx}{a \cos cx + b \sin cx} dx = \int \frac{dx}{a + b \tan cx} \\ = \frac{1}{c(a^2 + b^2)} [acx + b \log(a \cos cx + b \sin cx)]$$

$$348. \int \frac{\sin cx}{a \sin cx + b \cos cx} dx = \int \frac{dx}{a + b \cot cx} = \frac{1}{c(a^2 + b^2)} [acx - b \log(a \sin cx + b \cos cx)]$$

$$*349. \int \frac{dx}{a \cos^2 x + 2b \cos x \sin x + c \sin^2 x} = \begin{cases} \frac{1}{2\sqrt{b^2 - ac}} \log \frac{c \tan x + b - \sqrt{b^2 - ac}}{c \tan x + b + \sqrt{b^2 - ac}}, & (b^2 > ac) \\ \text{or} \\ \frac{1}{\sqrt{ac - b^2}} \tan^{-1} \frac{c \tan x + b}{\sqrt{ac - b^2}}, & (b^2 < ac) \\ \text{or} \\ -\frac{1}{c \tan x + b}, & (b^2 = ac) \end{cases}$$

$$350. \int \frac{\sin ax}{1 \pm \sin ax} dx = \pm x + \frac{1}{a} \tan \left( \frac{\pi}{4} \mp \frac{ax}{2} \right)$$

$$351. \int \frac{dx}{(\sin ax)(1 \pm \sin ax)} = \frac{1}{a} \tan \left( \frac{\pi}{4} \mp \frac{ax}{2} \right) + \frac{1}{a} \log \tan \frac{ax}{2}$$

$$352. \int \frac{dx}{(1 + \sin ax)^2} = -\frac{1}{2a} \tan \left( \frac{\pi}{4} - \frac{ax}{2} \right) - \frac{1}{6a} \tan^3 \left( \frac{\pi}{4} - \frac{ax}{2} \right)$$

$$353. \int \frac{dx}{(1 - \sin ax)^2} = \frac{1}{2a} \cot \left( \frac{\pi}{4} - \frac{ax}{2} \right) + \frac{1}{6a} \cot^3 \left( \frac{\pi}{4} - \frac{ax}{2} \right)$$

$$354. \int \frac{\sin ax}{(1 + \sin ax)^2} dx = -\frac{1}{2a} \tan \left( \frac{\pi}{4} - \frac{ax}{2} \right) + \frac{1}{6a} \tan^3 \left( \frac{\pi}{4} - \frac{ax}{2} \right)$$

$$355. \int \frac{\sin ax}{(1 - \sin ax)^2} dx = -\frac{1}{2a} \cot \left( \frac{\pi}{4} - \frac{ax}{2} \right) + \frac{1}{6a} \cot^3 \left( \frac{\pi}{4} - \frac{ax}{2} \right)$$

$$356. \int \frac{\sin x \, dx}{a + b \sin x} = \frac{x}{b} - \frac{a}{b} \int \frac{dx}{a + b \sin x}$$

$$357. \int \frac{dx}{(\sin x)(a + b \sin x)} = \frac{1}{a} \log \tan \frac{x}{2} - \frac{b}{a} \int \frac{dx}{a + b \sin x}$$

$$358. \int \frac{dx}{(a + b \sin x)^2} = \frac{b \cos x}{(a^2 - b^2)(a + b \sin x)} + \frac{a}{a^2 - b^2} \int \frac{dx}{a + b \sin x}$$

\*See note 6 on page A-19.

INTEGRALS (Continued)

$$359. \int \frac{\sin x \, dx}{(a + b \sin x)^2} = \frac{a \cos x}{(b^2 - a^2)(a + b \sin x)} + \frac{h}{b^2 - a^2} \int \frac{dx}{a + b \sin x}$$

$$*360. \int \frac{dx}{a^2 + b^2 \sin^2 cx} = \frac{1}{ac\sqrt{a^2 + b^2}} \tan^{-1} \frac{\sqrt{a^2 + b^2} \tan cx}{a}$$

$$*361. \int \frac{dx}{a^2 - b^2 \sin^2 cx} = \begin{cases} \frac{1}{ac\sqrt{a^2 - b^2}} \tan^{-1} \frac{\sqrt{a^2 - b^2} \tan cx}{a}, & (a^2 > b^2) \\ \text{or} \\ \frac{1}{2ac\sqrt{b^2 - a^2}} \log \frac{\sqrt{b^2 - a^2} \tan cx + a}{\sqrt{b^2 - a^2} \tan cx - a}, & (a^2 < b^2) \end{cases}$$

$$362. \int \frac{\cos ax}{1 + \cos ax} dx = x - \frac{1}{a} \tan \frac{ax}{2}$$

$$363. \int \frac{\cos ax}{1 - \cos ax} dx = -x - \frac{1}{a} \cot \frac{ax}{2}$$

$$364. \int \frac{dx}{(\cos ax)(1 + \cos ax)} = \frac{1}{a} \log \tan \left( \frac{\pi}{4} + \frac{ax}{2} \right) - \frac{1}{a} \tan \frac{ax}{2}$$

$$365. \int \frac{dx}{(\cos ax)(1 - \cos ax)} = \frac{1}{a} \log \tan \left( \frac{\pi}{4} + \frac{ax}{2} \right) - \frac{1}{a} \cot \frac{ax}{2}$$

$$366. \int \frac{dx}{(1 + \cos ax)^2} = \frac{1}{2a} \tan \frac{ax}{2} + \frac{1}{6a} \tan^3 \frac{ax}{2}$$

$$367. \int \frac{dx}{(1 - \cos ax)^2} = -\frac{1}{2a} \cot \frac{ax}{2} - \frac{1}{6a} \cot^3 \frac{ax}{2}$$

$$368. \int \frac{\cos ax}{(1 + \cos ax)^2} dx = \frac{1}{2a} \tan \frac{ax}{2} - \frac{1}{6a} \tan^3 \frac{ax}{2}$$

$$369. \int \frac{\cos ax}{(1 - \cos ax)^2} dx = \frac{1}{2a} \cot \frac{ax}{2} - \frac{1}{6a} \cot^3 \frac{ax}{2}$$

$$370. \int \frac{\cos x \, dx}{a + b \cos x} = \frac{x}{b} - \frac{a}{b} \int \frac{dx}{a + b \cos x}$$

$$371. \int \frac{dx}{(\cos x)(a + b \cos x)} = \frac{1}{a} \log \tan \left( \frac{x}{2} + \frac{\pi}{4} \right) - \frac{b}{a} \int \frac{dx}{a + b \cos x}$$

$$372. \int \frac{dx}{(a + b \cos x)^2} = \frac{b \sin x}{(b^2 - a^2)(a + b \cos x)} - \frac{a}{b^2 - a^2} \int \frac{dx}{a + b \cos x}$$

$$373. \int \frac{\cos x}{(a + b \cos x)^2} dx = \frac{a \sin x}{(a^2 - b^2)(a + b \cos x)} - \frac{b}{a^2 - b^2} \int \frac{dx}{a + b \cos x}$$

$$*374. \int \frac{dx}{a^2 + b^2 - 2ab \cos cx} = \frac{2}{c(a^2 - b^2)} \tan^{-1} \left( \frac{a + b}{a - b} \tan \frac{cx}{2} \right)$$

$$*375. \int \frac{dx}{a^2 + b^2 \cos^2 cx} = \frac{1}{ac\sqrt{a^2 + b^2}} \tan^{-1} \frac{a \tan cx}{\sqrt{a^2 + b^2}}$$

$$*376. \int \frac{dx}{a^2 - b^2 \cos^2 cx} = \begin{cases} \frac{1}{ac\sqrt{a^2 - b^2}} \tan^{-1} \frac{a \tan cx}{\sqrt{a^2 - b^2}}, & (a^2 > b^2) \\ \text{or} \\ \frac{1}{2ac\sqrt{b^2 - a^2}} \log \frac{a \tan cx - \sqrt{b^2 - a^2}}{a \tan cx + \sqrt{b^2 - a^2}}, & (b^2 > a^2) \end{cases}$$

$$377. \int \frac{\sin ax}{1 \pm \cos ax} dx = \mp \frac{1}{a} \log(1 \pm \cos ax)$$

\*See note 6 on page A-19.



INTEGRALS (Continued)

378.  $\int \frac{\cos ax}{1 \pm \sin ax} dx = \pm \frac{1}{a} \log(1 \pm \sin ax)$
379.  $\int \frac{dx}{(\sin ax)(1 \pm \cos ax)} = \pm \frac{1}{2a(1 \pm \cos ax)} + \frac{1}{2a} \log \tan \frac{ax}{2}$
380.  $\int \frac{dx}{(\cos ax)(1 \pm \sin ax)} = \mp \frac{1}{2a(1 \pm \sin ax)} + \frac{1}{2a} \log \tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)$
381.  $\int \frac{\sin ax}{(\cos ax)(1 \pm \cos ax)} dx = \frac{1}{a} \log(\sec ax \pm 1)$
382.  $\int \frac{\cos ax}{(\sin ax)(1 \pm \sin ax)} dx = -\frac{1}{a} \log(\csc ax \pm 1)$
383.  $\int \frac{\sin ax}{(\cos ax)(1 \pm \sin ax)} dx = \frac{1}{2a(1 \pm \sin ax)} \pm \frac{1}{2a} \log \tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)$
384.  $\int \frac{\cos ax}{(\sin ax)(1 \pm \cos ax)} dx = -\frac{1}{2a(1 \pm \cos ax)} \pm \frac{1}{2a} \log \tan \frac{ax}{2}$
385.  $\int \frac{dx}{\sin ax \pm \cos ax} = \frac{1}{a\sqrt{2}} \log \tan\left(\frac{ax}{2} \pm \frac{\pi}{8}\right)$
386.  $\int \frac{dx}{(\sin ax \pm \cos ax)^2} = \frac{1}{2a} \tan\left(ax \mp \frac{\pi}{4}\right)$
387.  $\int \frac{dx}{1 + \cos ax \pm \sin ax} = \pm \frac{1}{a} \log\left(1 \pm \tan \frac{ax}{2}\right)$
388.  $\int \frac{dx}{a^2 \cos^2 cx - b^2 \sin^2 cx} = \frac{1}{2abc} \log \frac{b \tan cx + a}{b \tan cx - a}$
389.  $\int x(\sin ax) dx = \frac{1}{a^2} \sin ax - \frac{x}{a} \cos ax$
390.  $\int x^2(\sin ax) dx = \frac{2x}{a^2} \sin ax - \frac{a^2 x^2 - 2}{a^3} \cos ax$
391.  $\int x^3(\sin ax) dx = \frac{3a^2 x^2 - 6}{a^4} \sin ax - \frac{a^2 x^3 - 6x}{a^3} \cos ax$
392.  $\int x^m \sin ax dx = \begin{cases} -\frac{1}{a} x^m \cos ax + \frac{m}{a} \int x^{m-1} \cos ax dx \\ \text{or} \\ \cos ax \sum_{r=0}^{[m/2]} (-1)^{r+1} \frac{m!}{(m-2r)!} \cdot \frac{x^{m-2r}}{a^{2r+1}} \\ + \sin ax \sum_{r=0}^{[(m-1)/2]} (-1)^r \frac{m!}{(m-2r-1)!} \cdot \frac{x^{m-2r-1}}{a^{2r+2}} \end{cases}$
- Note:  $[s]$  means greatest integer  $\leq s$ ;  $[3\frac{1}{2}] = 3$ ,  $[\frac{1}{2}] = 0$ , etc.
393.  $\int x(\cos ax) dx = \frac{1}{a^2} \cos ax + \frac{x}{a} \sin ax$
394.  $\int x^2(\cos ax) dx = \frac{2x \cos ax}{a^2} + \frac{a^2 x^2 - 2}{a^3} \sin ax$
395.  $\int x^3(\cos ax) dx = \frac{3a^2 x^2 - 6}{a^4} \cos ax + \frac{a^2 x^3 - 6x}{a^3} \sin ax$

INTEGRALS (Continued)

$$396. \int x^m (\cos ax) dx = \begin{cases} \frac{x^m \sin ax}{a} - \frac{m}{a} \int x^{m-1} \sin ax dx \\ \text{or} \\ \sin ax \sum_{r=0}^{[m/2]} (-1)^r \frac{m!}{(m-2r)!} \cdot \frac{x^{m-2r}}{a^{2r+1}} \\ + \cos ax \sum_{r=0}^{[(m-1)/2]} (-1)^r \frac{m!}{(m-2r-1)!} \cdot \frac{x^{m-2r-1}}{a^{2r+2}} \end{cases}$$

See note integral 392.

$$397. \int \frac{\sin ax}{x} dx = \sum_{n=0}^r (-1)^n \frac{(ax)^{2n+1}}{(2n+1)(2n+1)!}$$

$$398. \int \frac{\cos ax}{x} dx = \log x + \sum_{n=1}^r (-1)^n \frac{(ax)^{2n}}{2n(2n)!}$$

$$399. \int x(\sin^2 ax) dx = \frac{x^2}{4} - \frac{x \sin 2ax}{4a} - \frac{\cos 2ax}{8a^2}$$

$$400. \int x^2(\sin^2 ax) dx = \frac{x^3}{6} - \left(\frac{x^2}{4a} - \frac{1}{8a^3}\right) \sin 2ax - \frac{x \cos 2ax}{4a^2}$$

$$401. \int x(\sin^3 ax) dx = \frac{x \cos 3ax}{12a} - \frac{\sin 3ax}{36a^2} - \frac{3x \cos ax}{4a} + \frac{3 \sin ax}{4a^2}$$

$$402. \int x(\cos^2 ax) dx = \frac{x^2}{4} + \frac{x \sin 2ax}{4a} + \frac{\cos 2ax}{8a^2}$$

$$403. \int x^2(\cos^2 ax) dx = \frac{x^3}{6} + \left(\frac{x^2}{4a} - \frac{1}{8a^3}\right) \sin 2ax + \frac{x \cos 2ax}{4a^2}$$

$$404. \int x(\cos^3 ax) dx = \frac{x \sin 3ax}{12a} + \frac{\cos 3ax}{36a^2} + \frac{3x \sin ax}{4a} + \frac{3 \cos ax}{4a^2}$$

$$405. \int \frac{\sin ax}{x^m} dx = -\frac{\sin ax}{(m-1)x^{m-1}} + \frac{a}{m-1} \int \frac{\cos ax}{x^{m-1}} dx$$

$$406. \int \frac{\cos ax}{x^m} dx = -\frac{\cos ax}{(m-1)x^{m-1}} - \frac{a}{m-1} \int \frac{\sin ax}{x^{m-1}} dx$$

$$407. \int \frac{x}{1 \pm \sin ax} dx = \mp \frac{x \cos ax}{a(1 \pm \sin ax)} + \frac{1}{a^2} \log(1 \pm \sin ax)$$

$$408. \int \frac{x}{1 + \cos ax} dx = \frac{x}{a} \tan \frac{ax}{2} + \frac{2}{a^2} \log \cos \frac{ax}{2}$$

$$409. \int \frac{x}{1 - \cos ax} dx = -\frac{x}{a} \cot \frac{ax}{2} + \frac{2}{a^2} \log \sin \frac{ax}{2}$$

$$410. \int \frac{x + \sin x}{1 + \cos x} dx = x \tan \frac{x}{2}$$

$$411. \int \frac{x - \sin x}{1 - \cos x} dx = -x \cot \frac{x}{2}$$

$$412. \int \sqrt{1 - \cos ax} dx = -\frac{2 \sin ax}{a\sqrt{1 - \cos ax}} = -\frac{2\sqrt{2}}{a} \cos\left(\frac{ax}{2}\right)$$

$$413. \int \sqrt{1 + \cos ax} dx = \frac{2 \sin ax}{a\sqrt{1 + \cos ax}} = \frac{2\sqrt{2}}{a} \sin\left(\frac{ax}{2}\right)$$

$$414. \int \sqrt{1 + \sin x} dx = \pm 2 \left( \sin \frac{x}{2} - \cos \frac{x}{2} \right),$$

[use + if  $(8k-1)\frac{\pi}{2} < x \leq (8k+3)\frac{\pi}{2}$ , otherwise -;  $k$  an integer]

INTEGRALS (Continued)

415.  $\int \sqrt{1 - \sin x} \, dx = \pm 2 \left( \sin \frac{x}{2} + \cos \frac{x}{2} \right)$ ,  
 [use + if  $(8k - 3)\frac{\pi}{2} < x \leq (8k + 1)\frac{\pi}{2}$ , otherwise -;  $k$  an integer]
416.  $\int \frac{dx}{\sqrt{1 - \cos x}} = \pm \sqrt{2} \log \tan \frac{x}{4}$ ,  
 [use + if  $4k\pi < x < (4k + 2)\pi$ , otherwise -;  $k$  an integer]
417.  $\int \frac{dx}{\sqrt{1 + \cos x}} = \pm \sqrt{2} \log \tan \left( \frac{x + \pi}{4} \right)$ ,  
 [use + if  $(4k - 1)\pi < x < (4k + 1)\pi$ , otherwise -;  $k$  an integer]
418.  $\int \frac{dx}{\sqrt{1 - \sin x}} = \pm \sqrt{2} \log \tan \left( \frac{x}{4} - \frac{\pi}{8} \right)$ ,  
 [use + if  $(8k + 1)\frac{\pi}{2} < x < (8k + 5)\frac{\pi}{2}$ , otherwise -;  $k$  an integer]
419.  $\int \frac{dx}{\sqrt{1 + \sin x}} = \pm \sqrt{2} \log \tan \left( \frac{x}{4} + \frac{\pi}{8} \right)$ ,  
 [use + if  $(8k - 1)\frac{\pi}{2} < x < (8k + 3)\frac{\pi}{2}$ , otherwise -;  $k$  an integer]
420.  $\int (\tan^2 ax) \, dx = \frac{1}{a} \tan ax - x$
421.  $\int (\tan^3 ax) \, dx = \frac{1}{2a} \tan^2 ax + \frac{1}{a} \log \cos ax$
422.  $\int (\tan^4 ax) \, dx = \frac{\tan^3 ax}{3a} - \frac{1}{a} \tan x + x$
423.  $\int (\tan^n ax) \, dx = \frac{\tan^{n-1} ax}{a(n-1)} - \int (\tan^{n-2} ax) \, dx$
424.  $\int (\cot^2 ax) \, dx = -\frac{1}{a} \cot ax - x$
425.  $\int (\cot^3 ax) \, dx = -\frac{1}{2a} \cot^2 ax - \frac{1}{a} \log \sin ax$
426.  $\int (\cot^4 ax) \, dx = -\frac{1}{3a} \cot^3 ax + \frac{1}{a} \cot ax + x$
427.  $\int (\cot^n ax) \, dx = -\frac{\cot^{n-1} ax}{a(n-1)} - \int (\cot^{n-2} ax) \, dx$
428.  $\int \frac{x}{\sin^2 ax} \, dx = \int x(\csc^2 ax) \, dx = -\frac{x \cot ax}{a} + \frac{1}{a^2} \log \sin ax$
429.  $\int \frac{x}{\sin^n ax} \, dx = \int x(\csc^n ax) \, dx = -\frac{x \cos ax}{a(n-1)\sin^{n-1} ax} - \frac{1}{a^2(n-1)(n-2)\sin^{n-2} ax}$   
 $+ \frac{(n-2)}{(n-1)} \int \frac{x}{\sin^{n-2} ax} \, dx$
430.  $\int \frac{x}{\cos^2 ax} \, dx = \int x(\sec^2 ax) \, dx = \frac{1}{a} x \tan ax + \frac{1}{a^2} \log \cos ax$
431.  $\int \frac{x}{\cos^n ax} \, dx = \int x(\sec^n ax) \, dx = \frac{x \sin ax}{a(n-1) \cos^{n-1} ax} - \frac{1}{a^2(n-1)(n-2) \cos^{n-2} ax}$   
 $+ \frac{n-2}{n-1} \int \frac{x}{\cos^{n-2} ax} \, dx$

**INTEGRALS (Continued)**

432.  $\int \frac{\sin ax}{\sqrt{1+b^2 \sin^2 ax}} dx = -\frac{1}{ab} \sin^{-1} \frac{b \cos ax}{\sqrt{1+b^2}}$
433.  $\int \frac{\sin ax}{\sqrt{1-b^2 \sin^2 ax}} dx = -\frac{1}{ab} \log(b \cos ax + \sqrt{1-b^2 \sin^2 ax})$
434.  $\int (\sin ax) \sqrt{1+b^2 \sin^2 ax} dx = -\frac{\cos ax}{2a} \sqrt{1+b^2 \sin^2 ax} - \frac{1+b^2}{2ab} \sin^{-1} \frac{b \cos ax}{\sqrt{1+b^2}}$
435.  $\int (\sin ax) \sqrt{1-b^2 \sin^2 ax} dx = -\frac{\cos ax}{2a} \sqrt{1-b^2 \sin^2 ax} - \frac{1-b^2}{2ab} \log(b \cos ax + \sqrt{1-b^2 \sin^2 ax})$
436.  $\int \frac{\cos ax}{\sqrt{1+b^2 \sin^2 ax}} dx = \frac{1}{ab} \log(b \sin ax + \sqrt{1+b^2 \sin^2 ax})$
437.  $\int \frac{\cos ax}{\sqrt{1-b^2 \sin^2 ax}} dx = \frac{1}{ab} \sin^{-1}(b \sin ax)$
438.  $\int (\cos ax) \sqrt{1+b^2 \sin^2 ax} dx = \frac{\sin ax}{2a} \sqrt{1+b^2 \sin^2 ax} + \frac{1}{2ab} \log(b \sin ax + \sqrt{1+b^2 \sin^2 ax})$
439.  $\int (\cos ax) \sqrt{1-b^2 \sin^2 ax} dx = \frac{\sin ax}{2a} \sqrt{1-b^2 \sin^2 ax} + \frac{1}{2ab} \sin^{-1}(b \sin ax)$
440.  $\int \frac{dx}{\sqrt{a+b \tan^2 cx}} = \frac{\pm 1}{c\sqrt{a-b}} \sin^{-1} \left( \sqrt{\frac{a-b}{a}} \sin cx \right), \quad (a > |b|)$   
 [use + if  $(2k-1)\frac{\pi}{2} < x \leq (2k+1)\frac{\pi}{2}$ , otherwise -;  $k$  an integer]

**FORMS INVOLVING INVERSE TRIGONOMETRIC FUNCTIONS**

441.  $\int (\sin^{-1} ax) dx = x \sin^{-1} ax + \frac{\sqrt{1-a^2x^2}}{a}$
442.  $\int (\cos^{-1} ax) dx = x \cos^{-1} ax - \frac{\sqrt{1-a^2x^2}}{a}$
443.  $\int (\tan^{-1} ax) dx = x \tan^{-1} ax - \frac{1}{2a} \log(1+a^2x^2)$
444.  $\int (\cot^{-1} ax) dx = x \cot^{-1} ax + \frac{1}{2a} \log(1+a^2x^2)$
445.  $\int (\sec^{-1} ax) dx = x \sec^{-1} ax - \frac{1}{a} \log(ax + \sqrt{a^2x^2-1})$
446.  $\int (\csc^{-1} ax) dx = x \csc^{-1} ax + \frac{1}{a} \log(ax + \sqrt{a^2x^2-1})$
447.  $\int \left( \sin^{-1} \frac{x}{a} \right) dx = x \sin^{-1} \frac{x}{a} + \sqrt{a^2-x^2}, \quad (a > 0)$
448.  $\int \left( \cos^{-1} \frac{x}{a} \right) dx = x \cos^{-1} \frac{x}{a} - \sqrt{a^2-x^2}, \quad (a > 0)$

INTEGRALS (Continued)

$$449. \int \left( \tan^{-1} \frac{x}{a} \right) dx = x \tan^{-1} \frac{x}{a} - \frac{a}{2} \log(a^2 + x^2)$$

$$450. \int \left( \cot^{-1} \frac{x}{a} \right) dx = x \cot^{-1} \frac{x}{a} + \frac{a}{2} \log(a^2 + x^2)$$

$$451. \int x [\sin^{-1}(ax)] dx = \frac{1}{4a^2} [(2a^2x^2 - 1) \sin^{-1}(ax) + ax \sqrt{1 - a^2x^2}]$$

$$452. \int x [\cos^{-1}(ax)] dx = \frac{1}{4a^2} [(2a^2x^2 - 1) \cos^{-1}(ax) - ax \sqrt{1 - a^2x^2}]$$

$$453. \int x^n [\sin^{-1}(ax)] dx = \frac{x^{n+1}}{n+1} \sin^{-1}(ax) - \frac{a}{n+1} \int \frac{x^{n+1} dx}{\sqrt{1 - a^2x^2}}, \quad (n \neq -1)$$

$$454. \int x^n [\cos^{-1}(ax)] dx = \frac{x^{n+1}}{n+1} \cos^{-1}(ax) + \frac{a}{n+1} \int \frac{x^{n+1} dx}{\sqrt{1 - a^2x^2}}, \quad (n \neq -1)$$

$$455. \int x(\tan^{-1} ax) dx = \frac{1 + a^2x^2}{2a^2} \tan^{-1} ax - \frac{x}{2a}$$

$$456. \int x^n (\tan^{-1} ax) dx = \frac{x^{n+1}}{n+1} \tan^{-1} ax - \frac{a}{n+1} \int \frac{x^{n+1}}{1 + a^2x^2} dx$$

$$457. \int x(\cot^{-1} ax) dx = \frac{1 + a^2x^2}{2a^2} \cot^{-1} ax + \frac{x}{2a}$$

$$458. \int x^n (\cot^{-1} ax) dx = \frac{x^{n+1}}{n+1} \cot^{-1} ax + \frac{a}{n+1} \int \frac{x^{n+1}}{1 + a^2x^2} dx$$

$$459. \int \frac{\sin^{-1}(ax)}{x^2} dx = a \log \left( \frac{1 - \sqrt{1 - a^2x^2}}{x} \right) - \frac{\sin^{-1}(ax)}{x}$$

$$460. \int \frac{\cos^{-1}(ax)}{x^2} dx = -\frac{1}{x} \cos^{-1}(ax) + a \log \frac{1 + \sqrt{1 - a^2x^2}}{x}$$

$$461. \int \frac{\tan^{-1}(ax)}{x^2} dx = -\frac{1}{x} \tan^{-1}(ax) - \frac{a}{2} \log \frac{1 + a^2x^2}{x^2}$$

$$462. \int \frac{\cot^{-1} ax}{x^2} dx = -\frac{1}{x} \cot^{-1} ax - \frac{a}{2} \log \frac{x^2}{a^2x^2 + 1}$$

$$463. \int (\sin^{-1} ax)^2 dx = x(\sin^{-1} ax)^2 - 2x + \frac{2\sqrt{1 - a^2x^2}}{a} \sin^{-1} ax$$

$$464. \int (\cos^{-1} ax)^2 dx = x(\cos^{-1} ax)^2 - 2x - \frac{2\sqrt{1 - a^2x^2}}{a} \cos^{-1} ax$$

$$465. \int (\sin^{-1} ax)^n dx = \begin{cases} x(\sin^{-1} ax)^n + \frac{n\sqrt{1 - a^2x^2}}{a} (\sin^{-1} ax)^{n-1} - n(n-1) \int (\sin^{-1} ax)^{n-2} dx \\ \text{or} \\ \sum_{r=0}^{[n/2]} (-1)^r \frac{n!}{(n-2r)!} x(\sin^{-1} ax)^{n-2r} \\ \quad + \sum_{r=0}^{[(n-1)/2]} (-1)^r \frac{n! \sqrt{1 - a^2x^2}}{(n-2r-1)! a} (\sin^{-1} ax)^{n-2r-1} \end{cases}$$

Note:  $[s]$  means greatest integer  $\leq s$ . Thus  $[3.5]$  means 3;  $[5] = 5$ ,  $[\frac{5}{2}] = 0$ .

INTEGRALS (Continued)

$$466. \int (\cos^{-1} ax)^n dx = \begin{cases} x(\cos^{-1} ax)^n - \frac{n\sqrt{1-a^2x^2}}{a}(\cos^{-1} ax)^{n-1} - n(n-1) \int (\cos^{-1} ax)^{n-2} dx \\ \text{or} \\ \sum_{r=0}^{\lfloor n/2 \rfloor} (-1)^r \frac{n!}{(n-2r)!} x(\cos^{-1} ax)^{n-2r} \\ \sum_{r=0}^{\lfloor (n-1)/2 \rfloor} (-1)^r \frac{n!\sqrt{1-a^2x^2}}{(n-2r-1)!a} (\cos^{-1} ax)^{n-2r-1} \end{cases}$$

$$467. \int \frac{1}{\sqrt{1-a^2x^2}} (\sin^{-1} ax) dx = \frac{1}{2a} (\sin^{-1} ax)^2$$

$$468. \int \frac{x^n}{\sqrt{1-a^2x^2}} (\sin^{-1} ax) dx = -\frac{x^{n-1}}{na^2} \sqrt{1-a^2x^2} \sin^{-1} ax + \frac{x^n}{n^2a} + \frac{n-1}{na^2} \int \frac{x^{n-2}}{\sqrt{1-a^2x^2}} \sin^{-1} ax dx$$

$$469. \int \frac{1}{\sqrt{1-a^2x^2}} (\cos^{-1} ax) dx = -\frac{1}{2a} (\cos^{-1} ax)^2$$

$$470. \int \frac{x^n}{\sqrt{1-a^2x^2}} (\cos^{-1} ax) dx = -\frac{x^{n-1}}{na^2} \sqrt{1-a^2x^2} \cos^{-1} ax - \frac{x^n}{n^2a} + \frac{n-1}{na^2} \int \frac{x^{n-2}}{\sqrt{1-a^2x^2}} \cos^{-1} ax dx$$

$$471. \int \frac{\tan^{-1} ax}{a^2x^2+1} dx = \frac{1}{2a} (\tan^{-1} ax)^2$$

$$472. \int \frac{\cot^{-1} ax}{a^2x^2+1} dx = -\frac{1}{2a} (\cot^{-1} ax)^2$$

$$473. \int x \sec^{-1} ax dx = \frac{x^2}{2} \sec^{-1} ax - \frac{1}{2a^2} \sqrt{a^2x^2-1}$$

$$474. \int x^n \sec^{-1} ax dx = \frac{x^{n+1}}{n+1} \sec^{-1} ax - \frac{1}{n+1} \int \frac{x^n dx}{\sqrt{a^2x^2-1}}$$

$$475. \int \frac{\sec^{-1} ax}{x^2} dx = -\frac{\sec^{-1} ax}{x} + \frac{\sqrt{a^2x^2-1}}{x}$$

$$476. \int x \csc^{-1} ax dx = \frac{x^2}{2} \csc^{-1} ax + \frac{1}{2a^2} \sqrt{a^2x^2-1}$$

$$477. \int x^n \csc^{-1} ax dx = \frac{x^{n+1}}{n+1} \csc^{-1} ax + \frac{1}{n+1} \int \frac{x^n dx}{\sqrt{a^2x^2-1}}$$

$$478. \int \frac{\csc^{-1} ax}{x^2} dx = -\frac{\csc^{-1} ax}{x} - \frac{\sqrt{a^2x^2-1}}{x}$$

FORMS INVOLVING TRIGONOMETRIC SUBSTITUTIONS

$$479. \int f(\sin x) dx = 2 \int f\left(\frac{2z}{1+z^2}\right) \frac{dz}{1+z^2}, \quad \left(z = \tan \frac{x}{2}\right)$$

$$480. \int f(\cos x) dx = 2 \int f\left(\frac{1-z^2}{1+z^2}\right) \frac{dz}{1+z^2}, \quad \left(z = \tan \frac{x}{2}\right)$$

INTEGRALS (Continued)

$$*481. \int f(\sin x) dx = \int f(u) \frac{du}{\sqrt{1-u^2}}, \quad (u = \sin x)$$

$$*482. \int f(\cos x) dx = - \int f(u) \frac{du}{\sqrt{1-u^2}}, \quad (u = \cos x)$$

$$*483. \int f(\sin x, \cos x) dx = \int f(u, \sqrt{1-u^2}) \frac{du}{\sqrt{1-u^2}}, \quad (u = \sin x)$$

$$484. \int f(\sin x, \cos x) dx = 2 \int f\left(\frac{2z}{1+z^2}, \frac{1-z^2}{1+z^2}\right) \frac{dz}{1+z^2}, \quad \left(z = \tan \frac{x}{2}\right)$$

LOGARITHMIC FORMS

$$485. \int (\log x) dx = x \log x - x$$

$$486. \int x(\log x) dx = \frac{x^2}{2} \log x - \frac{x^2}{4}$$

$$487. \int x^2(\log x) dx = \frac{x^3}{3} \log x - \frac{x^3}{9}$$

$$488. \int x^n(\log ax) dx = \frac{x^{n+1}}{n+1} \log ax - \frac{x^{n+1}}{(n+1)^2}$$

$$489. \int (\log x)^2 dx = x(\log x)^2 - 2x \log x + 2x$$

$$490. \int (\log x)^n dx = \begin{cases} x(\log x)^n - n \int (\log x)^{n-1} dx, & (n \neq -1) \\ \text{or} \\ (-1)^n n! x \sum_{r=0}^n \frac{(-\log x)^r}{r!} \end{cases}$$

$$491. \int \frac{(\log x)^n}{x} dx = \frac{1}{n+1} (\log x)^{n+1}$$

$$492. \int \frac{dx}{\log x} = \log(\log x) + \log x + \frac{(\log x)^2}{2 \cdot 2!} + \frac{(\log x)^3}{3 \cdot 3!} + \dots$$

$$493. \int \frac{dx}{x \log x} = \log(\log x)$$

$$494. \int \frac{dx}{x(\log x)^n} = -\frac{1}{(n-1)(\log x)^{n-1}}$$

$$495. \int \frac{x^m dx}{(\log x)^n} = -\frac{x^{m+1}}{(n-1)(\log x)^{n-1}} + \frac{m+1}{n-1} \int \frac{x^m dx}{(\log x)^{n-1}}$$

$$496. \int x^m (\log x)^n dx = \begin{cases} \frac{x^{m+1}(\log x)^n}{m+1} - \frac{n}{m+1} \int x^m (\log x)^{n-1} dx \\ \text{or} \\ (-1)^n \frac{n!}{m+1} x^{m+1} \sum_{r=0}^n \frac{(-\log x)^r}{r!(m+1)^{n-r}} \end{cases}$$

$$497. \int x^p \cos(b \ln x) dx = \frac{x^{p+1}}{(p+1)^2 + b^2} \cdot [b \sin(b \ln x) + (p+1) \cos(b \ln x)] + c$$

$$498. \int x^p \sin(b \ln x) dx = \frac{x^{p+1}}{(p+1)^2 + b^2} \cdot [(p+1) \sin(b \ln x) - b \cos(b \ln x)] + c$$

$$499. \int [\log(ax+b)] dx = \frac{ax+b}{a} \log(ax+b) - x$$

\* The square roots appearing in these formulas may be plus or minus, depending on the quadrant of  $x$ . Care must be used to give them the proper sign.

INTEGRALS (Continued)

$$500. \int \frac{\log(ax+b)}{x^2} dx = \frac{a}{b} \log x - \frac{ax+b}{bx} \log(ax+b)$$

$$501. \int x^m [\log(ax+b)] dx = \frac{1}{m+1} \left[ x^{m+1} - \left(-\frac{b}{a}\right)^{m+1} \right] \log(ax+b) \\ - \frac{1}{m+1} \left(-\frac{b}{a}\right)^{m+1} \sum_{r=1}^{m+1} \frac{1}{r} \left(-\frac{ax}{b}\right)^r$$

$$502. \int \frac{\log(ax+b)}{x^m} dx = -\frac{1}{m-1} \frac{\log(ax+b)}{x^{m-1}} + \frac{1}{m-1} \left(-\frac{a}{b}\right)^{m-1} \log \frac{ax+b}{x} \\ + \frac{1}{m-1} \left(-\frac{a}{b}\right)^{m-1} \sum_{r=1}^{m-2} \frac{1}{r} \left(-\frac{b}{ax}\right)^r, \quad (m > 2)$$

$$503. \int \left[ \log \frac{x+a}{x-a} \right] dx = (x+a) \log(x+a) - (x-a) \log(x-a)$$

$$504. \int x^m \left[ \log \frac{x+a}{x-a} \right] dx = \frac{x^{m+1} - (-a)^{m+1}}{m+1} \log(x+a) - \frac{x^{m+1} - a^{m+1}}{m+1} \log(x-a) \\ + \frac{2a^{m+1} \sum_{r=1}^{[(m+1)/2]} 1}{m+1} \frac{1}{m-2r+2} \left(\frac{x}{a}\right)^{m-2r+2}$$

See note integral 392.

$$505. \int \frac{1}{x^2} \left[ \log \frac{x+a}{x-a} \right] dx = \frac{1}{x} \log \frac{x-a}{x+a} - \frac{1}{a} \log \frac{x^2-a^2}{x^2}$$

$$506. \int (\log X) dx = \begin{cases} \left(x + \frac{b}{2c}\right) \log X - 2x + \frac{\sqrt{4ac-b^2}}{c} \tan^{-1} \frac{2cx+b}{\sqrt{4ac-b^2}}, & (b^2-4ac < 0) \\ \text{or} \\ \left(x + \frac{b}{2c}\right) \log X - 2x + \frac{\sqrt{b^2-4ac}}{c} \tanh^{-1} \frac{2cx+b}{\sqrt{b^2-4ac}}, & (b^2-4ac > 0) \\ \text{where} \\ X = a + bx + cx^2 \end{cases}$$

$$507. \int x^n (\log X) dx = \frac{x^{n+1}}{n+1} \log X - \frac{2c}{n+1} \int \frac{x^{n+2}}{X} dx - \frac{b}{n+1} \int \frac{x^{n+1}}{X} dx \\ \text{where } X = a + bx + cx^2$$

$$508. \int [\log(x^2+a^2)] dx = x \log(x^2+a^2) - 2x + 2a \tan^{-1} \frac{x}{a}$$

$$509. \int [\log(x^2-a^2)] dx = x \log(x^2-a^2) - 2x + a \log \frac{x+a}{x-a}$$

$$510. \int x [\log(x^2 \pm a^2)] dx = \frac{1}{2}(x^2 \pm a^2) \log(x^2 \pm a^2) - \frac{1}{2}x^2$$

$$511. \int [\log(x + \sqrt{x^2 \pm a^2})] dx = x \log(x + \sqrt{x^2 \pm a^2}) - \sqrt{x^2 \pm a^2}$$

$$512. \int x [\log(x + \sqrt{x^2 \pm a^2})] dx = \left(\frac{x^2}{2} \pm \frac{a^2}{4}\right) \log(x + \sqrt{x^2 \pm a^2}) - \frac{x\sqrt{x^2 \pm a^2}}{4}$$

$$513. \int x^m [\log(x + \sqrt{x^2 \pm a^2})] dx = \frac{x^{m+1}}{m+1} \log(x + \sqrt{x^2 \pm a^2}) - \frac{1}{m+1} \int \frac{x^{m+1}}{\sqrt{x^2 \pm a^2}} dx$$



INTEGRALS (Continued)

$$514. \int \frac{\log(x + \sqrt{x^2 + a^2})}{x^2} dx = -\frac{\log(x + \sqrt{x^2 + a^2})}{x} - \frac{1}{a} \log \frac{a + \sqrt{x^2 + a^2}}{x}$$

$$515. \int \frac{\log(x + \sqrt{x^2 - a^2})}{x^2} dx = -\frac{\log(x + \sqrt{x^2 - a^2})}{x} + \frac{1}{|a|} \sec^{-1} \frac{x}{a}$$

$$516. \int x^n \log(x^2 - a^2) dx = \frac{1}{n+1} \left[ x^{n+1} \log(x^2 - a^2) - a^{n+1} \log(x - a) \right. \\ \left. - (-a)^{n+1} \log(x + a) - 2 \sum_{r=0}^{[n/2]} \frac{a^{2r} x^{n-2r+1}}{n-2r+1} \right]$$

See note integral 392.

EXPONENTIAL FORMS

$$517. \int e^x dx = e^x$$

$$518. \int e^{-x} dx = -e^{-x}$$

$$519. \int e^{ax} dx = \frac{e^{ax}}{a}$$

$$520. \int x e^{ax} dx = \frac{e^{ax}}{a^2} (ax - 1)$$

$$521. \int x^m e^{ax} dx = \begin{cases} \frac{x^m e^{ax}}{a} - \frac{m}{a} \int x^{m-1} e^{ax} dx \\ \text{or} \\ e^{ax} \sum_{r=0}^m (-1)^r \frac{m! x^{m-r}}{(m-r)! a^{r+1}} \end{cases}$$

$$522. \int \frac{e^{ax}}{x} dx = \log x + \frac{ax}{1!} + \frac{a^2 x^2}{2 \cdot 2!} + \frac{a^3 + x^3}{3 \cdot 3!} + \dots$$

$$523. \int \frac{e^{ax}}{x^m} dx = -\frac{1}{m-1} \frac{e^{ax}}{x^{m-1}} + \frac{a}{m-1} \int \frac{e^{ax}}{x^{m-1}} dx$$

$$524. \int e^{ax} \log x dx = \frac{e^{ax} \log x}{a} - \frac{1}{a} \int \frac{e^{ax}}{x} dx$$

$$525. \int \frac{dx}{1 + e^x} = x - \log(1 + e^x) = \log \frac{e^x}{1 + e^x}$$

$$526. \int \frac{dx}{a + be^{px}} = \frac{x}{a} - \frac{1}{ap} \log(a + be^{px})$$

$$527. \int \frac{dx}{ae^{mx} + be^{-mx}} = \frac{1}{m\sqrt{ab}} \tan^{-1} \left( e^{mx} \sqrt{\frac{a}{b}} \right), \quad (a > 0, b > 0)$$

$$528. \int \frac{dx}{ae^{mx} - be^{-mx}} = \begin{cases} \frac{1}{2m\sqrt{ab}} \log \frac{\sqrt{a} e^{mx} - \sqrt{b}}{\sqrt{a} e^{mx} + \sqrt{b}} \\ \text{or} \\ \frac{-1}{m\sqrt{ab}} \tanh^{-1} \left( \sqrt{\frac{a}{b}} e^{mx} \right), \quad (a > 0, b > 0) \end{cases}$$

$$529. \int (a^x - a^{-x}) dx = \frac{a^x + a^{-x}}{\log a}$$

$$530. \int \frac{e^{ax}}{b + ce^{ax}} dx = \frac{1}{ac} \log(b + ce^{ax})$$

$$531. \int \frac{x e^{ax}}{(1 + ax)^2} dx = \frac{e^{ax}}{a^2(1 + ax)}$$

INTEGRALS (Continued)

$$532. \int x e^{-x^2} dx = -\frac{1}{2}e^{-x^2}$$

$$533. \int e^{ax}[\sin(bx)] dx = \frac{e^{ax}[a \sin(bx) - b \cos(bx)]}{a^2 + b^2}$$

$$534. \int e^{ax}[\sin(bx)][\sin(cx)] dx = \frac{e^{ax}[(b-c) \sin(b-c)x + a \cos(b-c)x]}{2[a^2 + (b-c)^2]} - \frac{e^{ax}[(b+c) \sin(b+c)x + a \cos(b+c)x]}{2[a^2 + (b+c)^2]}$$

$$535. \int e^{ax}[\sin(bx)][\cos(cx)] dx = \begin{cases} \frac{e^{ax}[a \sin(b-c)x - (b-c) \cos(b-c)x]}{2[a^2 + (b-c)^2]} \\ + \frac{e^{ax}[a \sin(b+c)x - (b+c) \cos(b+c)x]}{2[a^2 + (b+c)^2]} \\ \text{or} \\ \frac{e^{ax}}{\rho} [(a \sin bx - b \cos bx) \cos(cx - \alpha) \\ - c(\sin bx) \sin(cx - \alpha)] \\ \text{where} \\ \rho = \sqrt{(a^2 + b^2 - c^2)^2 + 4a^2c^2}, \\ \rho \cos \alpha = a^2 + b^2 - c^2, \quad \rho \sin \alpha = 2ac \end{cases}$$

$$536. \int e^{ax}[\sin(bx)][\sin(bx+c)] dx = \frac{e^{ax} \cos c}{2a} - \frac{e^{ax}[a \cos(2bx+c) + 2b \sin(2bx+c)]}{2(a^2 + 4b^2)}$$

$$537. \int e^{ax}[\sin(bx)][\cos(bx+c)] dx = \frac{-e^{ax} \sin c}{2a} + \frac{e^{ax}[a \sin(2bx+c) - 2b \cos(2bx+c)]}{2(a^2 + 4b^2)}$$

$$538. \int e^{ax}[\cos(bx)] dx = \frac{e^{ax}}{a^2 + b^2} [a \cos(bx) + b \sin(bx)]$$

$$539. \int e^{ax}[\cos(bx)][\cos(cx)] dx = \frac{e^{ax}[(b-c) \sin(b-c)x + a \cos(b-c)x]}{2[a^2 + (b-c)^2]} + \frac{e^{ax}[(b+c) \sin(b+c)x + a \cos(b+c)x]}{2[a^2 + (b+c)^2]}$$

$$540. \int e^{ax}[\cos(bx)][\cos(bx+c)] dx = \frac{e^{ax} \cos c}{2a} + \frac{e^{ax}[a \cos(2bx+c) + 2b \sin(2bx+c)]}{2(a^2 + 4b^2)}$$

$$541. \int e^{ax}[\cos(bx)][\sin(bx+c)] dx = \frac{e^{ax} \sin c}{2a} + \frac{e^{ax}[a \sin(2bx+c) - 2b \cos(2bx+c)]}{2(a^2 + 4b^2)}$$

$$542. \int e^{ax}[\sin^n bx] dx = \frac{1}{a^2 + n^2b^2} \left[ (a \sin bx - nb \cos bx) e^{ax} \sin^{n-1} bx + n(n-1)b^2 \int e^{ax}[\sin^{n-2} bx] dx \right]$$

$$543. \int e^{ax}[\cos^n bx] dx = \frac{1}{a^2 + n^2b^2} \left[ (a \cos bx + nb \sin bx) e^{ax} \cos^{n-1} bx + n(n-1)b^2 \int e^{ax}[\cos^{n-2} bx] dx \right]$$

INTEGRALS (Continued)

$$544. \int x^m e^x \sin x \, dx = \frac{1}{2} x^m e^x (\sin x - \cos x) - \frac{m}{2} \int x^{m-1} e^x \sin x \, dx + \frac{m}{2} \int x^{m-1} e^x \cos x \, dx$$

$$545. \int x^m e^{ax} [\sin bx] \, dx = \begin{cases} x^m e^{ax} \frac{a \sin bx - b \cos bx}{a^2 + b^2} \\ - \frac{m}{a^2 + b^2} \int x^{m-1} e^{ax} (a \sin bx - b \cos bx) \, dx \\ \text{or} \\ e^{ax} \sum_{r=0}^m \frac{(-1)^r m! x^{m-r}}{\rho^{r+1} (m-r)!} \sin[bx - (r+1)\alpha] \\ \text{where} \\ \rho = \sqrt{a^2 + b^2}, \quad \rho \cos \alpha = a, \quad \rho \sin \alpha = b \end{cases}$$

$$546. \int x^m e^x \cos x \, dx = \frac{1}{2} x^m e^x (\sin x + \cos x) - \frac{m}{2} \int x^{m-1} e^x \sin x \, dx - \frac{m}{2} \int x^{m-1} e^x \cos x \, dx$$

$$547. \int x^m e^{ax} \cos bx \, dx = \begin{cases} x^m e^{ax} \frac{a \cos bx + b \sin bx}{a^2 + b^2} \\ - \frac{m}{a^2 + b^2} \int x^{m-1} e^{ax} (a \cos bx + b \sin bx) \, dx \\ \text{or} \\ e^{ax} \sum_{r=0}^m \frac{(-1)^r m! x^{m-r}}{\rho^{r+1} (m-r)!} \cos[bx - (r+1)\alpha] \\ \rho = \sqrt{a^2 + b^2}, \quad \rho \cos \alpha = a, \quad \rho \sin \alpha = b \end{cases}$$

INTEGRALS (Continued)

$$548. \int e^{ax}(\cos^m x)(\sin^n x) dx = \left\{ \begin{array}{l} \frac{e^{ax} \cos^{m-1} x \sin^n x [a \cos x + (m+n) \sin x]}{(m+n)^2 + a^2} \\ - \frac{na}{(m+n)^2 + a^2} \int e^{ax}(\cos^{m-1} x)(\sin^{n-1} x) dx \\ + \frac{(m-1)(m+n)}{(m+n)^2 + a^2} \int e^{ax}(\cos^{m-2} x)(\sin^n x) dx \\ \text{or} \\ \frac{e^{ax} \cos^m x [\sin^{n-1} x [a \sin x - (m+n) \cos x]}{(m+n)^2 + a^2} \\ + \frac{ma}{(m+n)^2 + a^2} \int e^{ax}(\cos^{m-1} x)(\sin^{n-1} x) dx \\ + \frac{(n-1)(m+n)}{(m+n)^2 + a^2} \int e^{ax}(\cos^m x)(\sin^{n-2} x) dx \\ \text{or} \\ \frac{e^{ax}(\cos^{m-1} x)(\sin^{n-1} x)(a \sin x \cos x + m \sin^2 x - n \cos^2 x)}{(m+n)^2 + a^2} \\ + \frac{m(m-1)}{(m+n)^2 + a^2} \int e^{ax}(\cos^{m-2} x)(\sin^n x) dx \\ + \frac{n(n-1)}{(m+n)^2 + a^2} \int e^{ax}(\cos^m x)(\sin^{n-2} x) dx \\ \text{or} \\ \frac{e^{ax}(\cos^{m-1} x)(\sin^{n-1} x)(a \cos x \sin x + m \sin^2 x - n \cos^2 x)}{(m+n)^2 + a^2} \\ + \frac{m(m-1)}{(m+n)^2 + a^2} \int e^{ax}(\cos^{m-2} x)(\sin^{n-2} x) dx \\ + \frac{(n-m)(n+m-1)}{(m+n)^2 + a^2} \int e^{ax}(\cos^m x)(\sin^{n-2} x) dx \end{array} \right.$$

$$549. \int x e^{ax}(\sin bx) dx = \frac{x e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx) - \frac{e^{ax}}{(a^2 + b^2)^2} [(a^2 - b^2) \sin bx - 2ab \cos bx]$$

$$550. \int x e^{ax}(\cos bx) dx = \frac{x e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx) - \frac{e^{ax}}{(a^2 + b^2)^2} [(a^2 - b^2) \cos bx + 2ab \sin bx]$$

$$551. \int \frac{e^{ax}}{\sin^n x} dx = -\frac{e^{ax}[a \sin x + (n-2) \cos x]}{(n-1)(n-2) \sin^{n-1} x} + \frac{a^2 + (n-2)^2}{(n-1)(n-2)} \int \frac{e^{ax}}{\sin^{n-2} x} dx$$

$$552. \int \frac{e^{ax}}{\cos^n x} dx = -\frac{e^{ax}[a \cos x - (n-2) \sin x]}{(n-1)(n-2) \cos^{n-1} x} + \frac{a^2 + (n-2)^2}{(n-1)(n-2)} \int \frac{e^{ax}}{\cos^{n-2} x} dx$$

$$553. \int e^{ax} \tan^n x dx = e^{ax} \frac{\tan^{n-1} x}{n-1} - \frac{a}{n-1} \int e^{ax} \tan^{n-1} x dx - \int e^{ax} \tan^{n-2} x dx$$

HYPERBOLIC FORMS

$$554. \int (\sinh x) dx = \cosh x$$

$$555. \int (\cosh x) dx = \sinh x$$

INTEGRALS (Continued)

$$556. \int (\tanh x) dx = \log \cosh x$$

$$557. \int (\coth x) dx = \log \sinh x$$

$$558. \int (\operatorname{sech} x) dx = \tan^{-1}(\sinh x)$$

$$559. \int (\operatorname{csch} x) dx = \log \tanh\left(\frac{x}{2}\right)$$

$$560. \int x(\sinh x) dx = x \cosh x - \sinh x$$

$$561. \int x^n(\sinh x) dx = x^n \cosh x - n \int x^{n-1}(\cosh x) dx$$

$$562. \int x(\cosh x) dx = x \sinh x - \cosh x$$

$$563. \int x^n(\cosh x) dx = x^n \sinh x - n \int x^{n-1}(\sinh x) dx$$

$$564. \int (\operatorname{sech} x)(\tanh x) dx = -\operatorname{sech} x$$

$$565. \int (\operatorname{csch} x)(\coth x) dx = -\operatorname{csch} x$$

$$566. \int (\sinh^2 x) dx = \frac{\sinh 2x}{4} - \frac{x}{2}$$

$$567. \int (\sinh^m x)(\cosh^n x) dx = \begin{cases} \frac{1}{m+n}(\sinh^{m+1} x)(\cosh^{n-1} x) \\ + \frac{n-1}{m+n} \int (\sinh^m x)(\cosh^{n-2} x) dx \\ \text{or} \\ \frac{1}{m+n} \sinh^{m-1} x \cosh^{n+1} x \\ - \frac{m-1}{m+n} \int (\sinh^{m-2} x)(\cosh^n x) dx, \quad (m+n \neq 0) \end{cases}$$

$$568. \int \frac{dx}{(\sinh^m x)(\cosh^n x)} \begin{cases} -\frac{1}{(m-n)(\sinh^{m-1} x)(\cosh^{n-1} x)} \\ -\frac{m+n-2}{m-1} \int \frac{dx}{(\sinh^{m-2} x)(\cosh^n x)}, \quad (m \neq 1) \\ \text{or} \\ \frac{1}{(n-1) \sinh^{m-1} x \cosh^{n-1} x} \\ + \frac{m+n-2}{n-1} \int \frac{dx}{(\sinh^m x)(\cosh^{n-2} x)}, \quad (n \neq 1) \end{cases}$$

$$569. \int (\tanh^2 x) dx = x - \tanh x$$

$$570. \int (\tanh^n x) dx = -\frac{\tanh^{n-1} x}{n-1} + \int (\tanh^{n-2} x) dx, \quad (n \neq 1)$$

$$571. \int (\operatorname{sech}^2 x) dx = \tanh x$$

$$572. \int (\cosh^2 x) dx = \frac{\sinh 2x}{4} + \frac{x}{2}$$

INTEGRALS (Continued)

573.  $\int (\coth^2 x) dx = x - \coth x$
574.  $\int (\coth^n x) dx = -\frac{\coth^{n-1} x}{n-1} + \int \coth^{n-2} x dx, \quad (n \neq 1)$
575.  $\int (\operatorname{csch}^2 x) dx = -\operatorname{ctnh} x$
576.  $\int (\sinh mx)(\sinh nx) dx = \frac{\sinh(m+n)x}{2(m+n)} - \frac{\sinh(m-n)x}{2(m-n)}, \quad (m^2 \neq n^2)$
577.  $\int (\cosh mx)(\cosh nx) dx = \frac{\sinh(m+n)x}{2(m+n)} + \frac{\sinh(m-n)x}{2(m-n)}, \quad (m^2 \neq n^2)$
578.  $\int (\sinh mx)(\cosh nx) dx = \frac{\cosh(m+n)x}{2(m+n)} + \frac{\cosh(m-n)x}{2(m-n)}, \quad (m^2 \neq n^2)$
579.  $\int (\sinh^{-1} \frac{x}{a}) dx = x \sinh^{-1} \frac{x}{a} - \sqrt{x^2 + a^2}, \quad (a > 0)$
580.  $\int x(\sinh^{-1} \frac{x}{a}) dx = (\frac{x^2}{2} + \frac{a^2}{4}) \sinh^{-1} \frac{x}{a} - \frac{x}{4} \sqrt{x^2 + a^2}, \quad (a > 0)$
581.  $\int x^n (\sinh^{-1} x) dx = (\frac{x^{n+1}}{n+1}) \sinh^{-1} x - \frac{1}{n+1} \int \frac{x^{n+1}}{(1+x^2)^{1/2}} dx, \quad (n \neq -1)$
582.  $\int (\cosh^{-1} \frac{x}{a}) dx = \begin{cases} x \cosh^{-1} \frac{x}{a} - \sqrt{x^2 - a^2}, & (\cosh^{-1} \frac{x}{a} > 0) \\ \text{or} \\ x \cosh^{-1} \frac{x}{a} + \sqrt{x^2 - a^2}, & (\cosh^{-1} \frac{x}{a} < 0), \end{cases} \quad (a > 0)$
583.  $\int x(\cosh^{-1} \frac{x}{a}) dx = \frac{2x^2 - a^2}{4} \cosh^{-1} \frac{x}{a} - \frac{x}{4} (x^2 - a^2)^{\frac{1}{2}}$
584.  $\int x^n (\cosh^{-1} x) dx = \frac{x^{n+1}}{n+1} \cosh^{-1} x - \frac{1}{n+1} \int \frac{x^{n+1}}{(x^2 - 1)^{1/2}} dx, \quad (n \neq -1)$
585.  $\int (\tanh^{-1} \frac{x}{a}) dx = x \tanh^{-1} \frac{x}{a} + \frac{a}{2} \log(a^2 - x^2), \quad (|\frac{x}{a}| < 1)$
586.  $\int (\coth^{-1} \frac{x}{a}) dx = x \coth^{-1} \frac{x}{a} + \frac{a}{2} \log(x^2 - a^2), \quad (|\frac{x}{a}| > 1)$
587.  $\int x(\tanh^{-1} \frac{x}{a}) dx = \frac{x^2 - a^2}{2} \tanh^{-1} \frac{x}{a} + \frac{ax}{2}, \quad (|\frac{x}{a}| < 1)$
588.  $\int x^n (\tanh^{-1} x) dx = \frac{x^{n+1}}{n+1} \tanh^{-1} x - \frac{1}{n+1} \int \frac{x^{n+1}}{1-x^2} dx, \quad (n \neq -1)$
589.  $\int x(\coth^{-1} \frac{x}{a}) dx = \frac{x^2 - a^2}{2} \coth^{-1} \frac{x}{a} + \frac{ax}{2}, \quad (|\frac{x}{a}| > 1)$
590.  $\int x^n (\coth^{-1} x) dx = \frac{x^{n+1}}{n+1} \coth^{-1} x + \frac{1}{n+1} \int \frac{x^{n+1}}{x^2 - 1} dx, \quad (n \neq -1)$
591.  $\int (\operatorname{sech}^{-1} x) dx = x \operatorname{sech}^{-1} x + \sin^{-1} x$
592.  $\int x \operatorname{sech}^{-1} x dx = \frac{x^2}{2} \operatorname{sech}^{-1} x - \frac{1}{2} \sqrt{1 - x^2}$
593.  $\int x^n \operatorname{sech}^{-1} x dx = \frac{x^{n+1}}{n+1} \operatorname{sech}^{-1} x + \frac{1}{n+1} \int \frac{x^n}{(1-x^2)^{1/2}} dx, \quad (n \neq -1)$
594.  $\int \operatorname{csch}^{-1} x dx = x \operatorname{csch}^{-1} x + \frac{x}{|x|} \sinh^{-1} x$

INTEGRALS (Continued)

$$595. \int x \operatorname{csch}^{-1} x \, dx = \frac{x^2}{2} \operatorname{csch}^{-1} x + \frac{1}{2} \frac{x}{|x|} \sqrt{1+x^2}$$

$$596. \int x^n \operatorname{csch}^{-1} x \, dx = \frac{x^{n+1}}{n+1} \operatorname{csch}^{-1} x + \frac{1}{n+1} \frac{x}{|x|} \int \frac{x^n}{(x^2+1)^{\frac{1}{2}}} dx, \quad (n \neq -1)$$

DEFINITE INTEGRALS

$$597. \int_0^\infty x^{n-1} e^{-x} dx = \int_0^1 \left(\log \frac{1}{x}\right)^{n-1} dx = \frac{1}{n} \prod_{m=1}^\infty \frac{\left(1 + \frac{1}{m}\right)^n}{1 + \frac{n}{m}}$$

=  $\Gamma(n)$ ,  $n \neq 0, -1, -2, -3, \dots$  (Gamma Function)

$$598. \int_0^\infty t^n p^{-t} dt = \frac{n!}{(\log p)^{n+1}}, \quad (n = 0, 1, 2, 3, \dots \text{ and } p > 0)$$

$$599. \int_0^\infty t^{n-1} e^{-(a+1)t} dt = \frac{\Gamma(n)}{(a+1)^n}, \quad (n > 0, a > -1)$$

$$600. \int_0^1 x^m \left(\log \frac{1}{x}\right)^n dx = \frac{\Gamma(n+1)}{(m+1)^{n+1}}, \quad (m > -1, n > -1)$$

601.  $\Gamma(n)$  is finite if  $n > 0$ ,  $\Gamma(n+1) = n\Gamma(n)$

602.  $\Gamma(n) \cdot \Gamma(1-n) = \frac{\pi}{\sin n\pi}$

603.  $\Gamma(n) = (n-1)!$  if  $n = \text{integer} > 0$

604.  $\Gamma\left(\frac{1}{2}\right) = 2 \int_0^\infty e^{-t^2} dt = \sqrt{\pi} = 1.7724538509 \dots = \left(-\frac{1}{2}\right)!$

605.  $\Gamma\left(n + \frac{1}{2}\right) = \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{2^n} \sqrt{\pi} \quad n = 1, 2, 3, \dots$

606.  $\Gamma\left(-n + \frac{1}{2}\right) = \frac{(-1)^n 2^n \sqrt{\pi}}{1 \cdot 3 \cdot 5 \dots (2n-1)} \quad n = 1, 2, 3, \dots$

$$607. \int_0^1 x^{m-1} (1-x)^{n-1} dx = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = B(m, n)$$

(Beta function)

608.  $B(m, n) = B(n, m) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ , where  $m$  and  $n$  are any positive real numbers.

$$609. \int_a^b (x-a)^m (b-x)^n dx = (b-a)^{m+n+1} \frac{\Gamma(m+1) \cdot \Gamma(n+1)}{\Gamma(m+n+2)}, \quad (m > -1, n > -1, b > a)$$

$$610. \int_1^\infty \frac{dx}{x^m} = \frac{1}{m-1}, \quad [m > 1]$$

$$611. \int_0^\infty \frac{dx}{(1+x)x^p} = \pi \csc p\pi, \quad [p < 1]$$

$$612. \int_0^\infty \frac{dx}{(1-x)x^p} = -\pi \cot p\pi, \quad [p < 1]$$

$$613. \int_0^\infty \frac{x^{p-1} dx}{(1+x)} = \frac{\pi}{\sin p\pi}$$

=  $B(p, 1-p) = \Gamma(p)\Gamma(1-p)$ ,  $[0 < p < 1]$

$$614. \int_0^\infty \frac{x^{m-1} dx}{1+x^n} = \frac{\pi}{n \sin \frac{m\pi}{n}}, \quad [0 < m < n]$$

**DEFINITE INTEGRALS (Continued)**

615. 
$$\int_0^\infty \frac{x^a dx}{(m+x^b)^c} = \frac{m^{(a+1-bc)/b}}{b} \left[ \frac{\Gamma\left(\frac{a+1}{b}\right)\Gamma\left(c-\frac{a+1}{b}\right)}{\Gamma(c)} \right]$$
  
 ( $a > -1, b > 0, m > 0, c > \frac{a+1}{b}$ )
616. 
$$\int_0^\infty \frac{dx}{(1+x)\sqrt{x}} = \pi$$
617. 
$$\int_0^\infty \frac{a dx}{a^2+x^2} = \frac{\pi}{2}, \quad \text{if } a > 0; 0, \text{ if } a = 0; -\frac{\pi}{2}, \text{ if } a < 0$$
618. 
$$\int_0^a (a^2-x^2)^{n/2} dx = \frac{1}{2} \int_{-a}^a (a^2-x^2)^{n/2} dx = \frac{1 \cdot 3 \cdot 5 \dots n}{2 \cdot 4 \cdot 6 \dots (n+1)} \cdot \frac{\pi}{2} \cdot a^{n+1} \quad (n \text{ odd})$$
619. 
$$\int_0^a x^m (a^2-x^2)^{n/2} dx = \begin{cases} \frac{1}{2} a^{m+n+1} B\left(\frac{m+1}{2}, \frac{n+2}{2}\right) \\ \text{or} \\ \frac{1}{2} a^{m+n+1} \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+2}{2}\right)}{\Gamma\left(\frac{m+n+3}{2}\right)} \end{cases}$$
620. 
$$\int_0^{\pi/2} (\sin^n x) dx = \begin{cases} \int_0^{\pi/2} (\cos^n x) dx \\ \text{or} \\ \frac{1 \cdot 3 \cdot 5 \cdot 7 \dots (n-1)\pi}{2 \cdot 4 \cdot 6 \cdot 8 \dots (n)} \cdot \frac{\pi}{2}, & (n \text{ an even integer, } n \neq 0) \\ \text{or} \\ \frac{2 \cdot 4 \cdot 6 \cdot 8 \dots (n-1)}{1 \cdot 3 \cdot 5 \cdot 7 \dots (n)}, & (n \text{ an odd integer, } n \neq 1) \\ \text{or} \\ \frac{\sqrt{\pi} \Gamma\left(\frac{n+1}{2}\right)}{2 \Gamma\left(\frac{n}{2}+1\right)}, & (n > -1) \end{cases}$$
621. 
$$\int_0^\infty \frac{\sin mx dx}{x} = \frac{\pi}{2}, \quad \text{if } m > 0; 0, \text{ if } m = 0; -\frac{\pi}{2}, \text{ if } m < 0$$
622. 
$$\int_0^\infty \frac{\cos x dx}{x} = \infty$$
623. 
$$\int_0^\infty \frac{\tan x dx}{x} = \frac{\pi}{2}$$
624. 
$$\int_0^\pi \sin ax \cdot \sin bx dx = \int_0^\pi \cos ax \cdot \cos bx dx = 0, \quad (a \neq b; a, b \text{ integers})$$
625. 
$$\int_0^{\pi/a} [\sin(ax)][\cos(ax)] dx = \int_0^\pi [\sin(ax)][\cos(ax)] dx = 0$$
626. 
$$\int_0^\pi [\sin(ax)][\cos(bx)] dx = \frac{2a}{a^2-b^2}, \text{ if } a-b \text{ is odd, or } 0 \text{ if } a-b \text{ is even}$$
627. 
$$\int_0^\infty \frac{\sin x \cos mx dx}{x} = 0, \quad \text{if } m < -1 \text{ or } m > 1; \frac{\pi}{4}, \text{ if } m = \pm 1; \frac{\pi}{2}, \text{ if } m^2 < 1$$



DEFINITE INTEGRALS (Continued)

628.  $\int_0^\infty \frac{\sin ax \sin bx}{x^2} dx = \frac{\pi a}{2}, \quad (a \leq b)$
629.  $\int_0^\pi \sin^2 mx \, dx = \int_0^\pi \cos^2 mx \, dx = \frac{\pi}{2}$
630.  $\int_0^\infty \frac{\sin^2(px)}{x^2} dx = \frac{\pi p}{2}$
631.  $\int_0^\infty \frac{\sin x}{x^p} dx = \frac{\pi}{2\Gamma(p)\sin(p\pi/2)}, \quad 0 < p < 1$
632.  $\int_0^\infty \frac{\cos x}{x^p} dx = \frac{\pi}{2\Gamma(p)\cos(p\pi/2)}, \quad 0 < p < 1$
633.  $\int_0^\infty \frac{1 - \cos px}{x^2} dx = \frac{\pi p}{2}$
634.  $\int_0^\infty \frac{\sin px \cos qx}{x} dx = \left\{ 0, q > p > 0; \frac{\pi}{2}, p > q > 0; \frac{\pi}{4}, p = q > 0 \right\}$
635.  $\int_0^\infty \frac{\cos(mx)}{x^2 + a^2} dx = \frac{\pi}{2|a|} e^{-|ma|}$
636.  $\int_0^\infty \cos(x^2) dx = \int_0^\infty \sin(x^2) dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$
637.  $\int_0^\infty \sin ax^n dx = \frac{1}{na^{1/n}} \Gamma(1/n) \sin \frac{\pi}{2n}, \quad n > 1$
638.  $\int_0^\infty \cos ax^n dx = \frac{1}{na^{1/n}} \Gamma(1/n) \cos \frac{\pi}{2n}, \quad n > 1$
639.  $\int_0^\infty \frac{\sin x}{\sqrt{x}} dx = \int_0^\infty \frac{\cos x}{\sqrt{x}} dx = \sqrt{\frac{\pi}{2}}$
640. (a)  $\int_0^\infty \frac{\sin^3 x}{x} dx = \frac{\pi}{4}$       (b)  $\int_0^\infty \frac{\sin^3 x}{x^2} dx = \frac{3}{4} \log 3$
641.  $\int_0^\infty \frac{\sin^3 x}{x^3} dx = \frac{3\pi}{8}$
642.  $\int_0^\infty \frac{\sin^4 x}{x^4} dx = \frac{\pi}{3}$
643.  $\int_0^{\pi/2} \frac{dx}{1 + a \cos x} = \frac{\cos^{-1} a}{\sqrt{1 - a^2}}, \quad (a < 1)$
644.  $\int_0^\pi \frac{dx}{a + b \cos x} = \frac{\pi}{\sqrt{a^2 - b^2}}, \quad (a > b \geq 0)$
645.  $\int_0^{2\pi} \frac{dx}{1 + a \cos x} = \frac{2\pi}{\sqrt{1 - a^2}}, \quad (a^2 < 1)$
646.  $\int_0^\infty \frac{\cos ax - \cos bx}{x} dx = \log \frac{b}{a}$
647.  $\int_0^{\pi/2} \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x} = \frac{\pi}{2ab}$
648.  $\int_0^{\pi/2} \frac{dx}{(a^2 \sin^2 x + b^2 \cos^2 x)^2} = \frac{\pi(a^2 + b^2)}{4a^3 b^3}, \quad (a, b > 0)$
649.  $\int_0^{\pi/2} \sin^{n-1} x \cos^{m-1} x \, dx = \frac{1}{2} B\left(\frac{n}{2}, \frac{m}{2}\right), \quad m \text{ and } n \text{ positive integers}$

**DEFINITE INTEGRALS (Continued)**

$$650. \int_0^{\pi/2} (\sin^{2n+1} \theta) d\theta = \frac{2 \cdot 4 \cdot 6 \dots (2n)}{1 \cdot 3 \cdot 5 \dots (2n+1)}, \quad (n = 1, 2, 3, \dots)$$

$$651. \int_0^{\pi/2} (\sin^{2n} \theta) d\theta = \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{2 \cdot 4 \dots (2n)} \left(\frac{\pi}{2}\right), \quad (n = 1, 2, 3, \dots)$$

$$652. \int_0^{\pi/2} \frac{x}{\sin x} dx = 2 \left\{ \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \dots \right\}$$

$$653. \int_0^{\pi/2} \frac{dx}{1 + \tan^m x} = \frac{\pi}{4}$$

$$654. \int_0^{\pi/2} \sqrt{\cos \theta} d\theta = \frac{(2\pi)^{\frac{3}{2}}}{[\Gamma(\frac{3}{4})]^2}$$

$$655. \int_0^{\pi/2} (\tan^h \theta) d\theta = \frac{\pi}{2 \cos\left(\frac{h\pi}{2}\right)}, \quad (0 < h < 1)$$

$$656. \int_0^{\infty} \frac{\tan^{-1}(ax) - \tan^{-1}(bx)}{x} dx = \frac{\pi}{2} \log \frac{a}{b}, \quad (a, b > 0)$$

657. The area enclosed by a curve defined through the equation  $x^{\frac{a}{c}} + y^{\frac{b}{c}} = a^{\frac{b}{c}}$  where  $a > 0$ ,  $c$  a positive odd integer and  $b$  a positive even integer is given by

$$\frac{[\Gamma(\frac{c}{b})]^2}{\Gamma(\frac{2c}{b})} \left(\frac{2ca^2}{b}\right)$$

658.  $I = \iiint_R x^{h-1} y^{m-1} z^{n-1} dv$ , where  $R$  denotes the region of space bounded by the

co-ordinate planes and that portion of the surface  $\left(\frac{x}{a}\right)^p + \left(\frac{y}{b}\right)^q + \left(\frac{z}{c}\right)^k = 1$ , which lies in the first octant and where  $h, m, n, p, q, k, a, b, c$ , denote positive real numbers is given by

$$\int_0^a x^{h-1} dx \int_0^{a[1-(x/a)^p]^{1/q}} y^{m-1} dy \int_0^{c[1-(x/a)^p - (y/b)^q]^{1/k}} z^{n-1} dz = \frac{a^h b^m c^n}{pqk} \frac{\Gamma\left(\frac{h}{p}\right)\Gamma\left(\frac{m}{q}\right)\Gamma\left(\frac{n}{k}\right)}{\Gamma\left(\frac{h}{p} + \frac{m}{q} + \frac{n}{k} + 1\right)}$$

$$659. \int_0^{\infty} e^{-ax} dx = \frac{1}{a}, \quad (a > 0)$$

$$660. \int_0^{\infty} \frac{e^{-ax} - e^{-bx}}{x} dx = \log \frac{b}{a}, \quad (a, b > 0)$$

$$661. \int_0^{\infty} x^n e^{-ax} dx = \begin{cases} \frac{\Gamma(n+1)}{a^{n+1}}, & (n > -1, a > 0) \\ \text{or} \\ \frac{n!}{a^{n+1}}, & (a > 0, n \text{ positive integer}) \end{cases}$$

$$662. \int_0^{\infty} x^n \exp(-ax^p) dx = \frac{\Gamma(k)}{pa^k}, \quad \left(n > -1, p > 0, a > 0, k = \frac{n+1}{p}\right)$$

$$663. \int_0^{\infty} e^{-a^2 x^2} dx = \frac{1}{2a} \sqrt{\pi} = \frac{1}{2a} \Gamma\left(\frac{1}{2}\right), \quad (a > 0)$$

$$664. \int_0^{\infty} x e^{-x^2} dx = \frac{1}{2}$$

**DEFINITE INTEGRALS (Continued)**

665.  $\int_0^{\infty} x^2 e^{-x^2} dx = \frac{\sqrt{\pi}}{4}$
666.  $\int_0^{\infty} x^{2n} e^{-ax^2} dx = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^{n+1} a^n} \sqrt{\frac{\pi}{a}}$
667.  $\int_0^{\infty} x^{2n+1} e^{-ax^2} dx = \frac{n!}{2a^{n+1}}, \quad (a > 0)$
668.  $\int_0^1 x^m e^{-ax} dx = \frac{m!}{a^{m+1}} \left[ 1 - e^{-a} \sum_{r=0}^m \frac{a^r}{r!} \right]$
669.  $\int_0^{\infty} e^{(-x^2 - a^2/x^2)} dx = \frac{e^{-2a} \sqrt{\pi}}{2}, \quad (a \geq 0)$
670.  $\int_0^{\infty} e^{-nx} \sqrt{x} dx = \frac{1}{2n} \sqrt{\frac{\pi}{n}}$
671.  $\int_0^{\infty} \frac{e^{-nx}}{\sqrt{x}} dx = \sqrt{\frac{\pi}{n}}$
672.  $\int_0^{\infty} e^{-ax} (\cos mx) dx = \frac{a}{a^2 + m^2}, \quad (a > 0)$
673.  $\int_0^{\infty} e^{-ax} (\sin mx) dx = \frac{m}{a^2 + m^2}, \quad (a > 0)$
674.  $\int_0^{\infty} x e^{-ax} [\sin(bx)] dx = \frac{2ab}{(a^2 + b^2)^2}, \quad (a > 0)$
675.  $\int_0^{\infty} x e^{-ax} [\cos(bx)] dx = \frac{a^2 - b^2}{(a^2 + b^2)^2}, \quad (a > 0)$
676.  $\int_0^{\infty} x^n e^{-ax} [\sin(bx)] dx = \frac{n![(a+ib)^{n+1} - (a-ib)^{n+1}]}{2i(a^2 + b^2)^{n+1}}, \quad (i^2 = -1, a > 0)$
677.  $\int_0^{\infty} x^n e^{-ax} [\cos(bx)] dx = \frac{n![(a-ib)^{n+1} + (a+ib)^{n+1}]}{2(a^2 + b^2)^{n+1}}, \quad (i^2 = -1, a > 0)$
678.  $\int_0^{\infty} \frac{e^{-ax} \sin x}{x} dx = \cot^{-1} a, \quad (a > 0)$
679.  $\int_0^{\infty} e^{-a^2 x^2} \cos bx dx = \frac{\sqrt{\pi}}{2a} \exp\left(-\frac{b^2}{4a^2}\right), \quad (ab \neq 0)$
680.  $\int_0^{\infty} e^{-t \cos \phi} t^{b-1} [\sin(t \sin \phi)] dt = [\Gamma(b)] \sin(b\phi), \quad \left(b > 0, -\frac{\pi}{2} < \phi < \frac{\pi}{2}\right)$
681.  $\int_0^{\infty} e^{-t \cos \phi} t^{b-1} [\cos(t \sin \phi)] dt = [\Gamma(b)] \cos(b\phi), \quad \left(b > 0, -\frac{\pi}{2} < \phi < \frac{\pi}{2}\right)$
682.  $\int_0^{\infty} t^{b-1} \cos t dt = [\Gamma(b)] \cos\left(\frac{b\pi}{2}\right), \quad (0 < b < 1)$
683.  $\int_0^{\infty} t^{b-1} (\sin t) dt = [\Gamma(b)] \sin\left(\frac{b\pi}{2}\right), \quad (0 < b < 1)$
684.  $\int_0^1 (\log x)^n dx = (-1)^n \cdot n!$
685.  $\int_0^1 \left(\log \frac{1}{x}\right)^{\frac{1}{2}} dx = \frac{\sqrt{\pi}}{2}$
686.  $\int_0^1 \left(\log \frac{1}{x}\right)^{-\frac{1}{2}} dx = \sqrt{\pi}$

**DEFINITE INTEGRALS (Continued)**

687.  $\int_0^1 \left(\log \frac{1}{x}\right)^n dx = n!$
688.  $\int_0^1 x \log(1-x) dx = -\frac{3}{4}$
689.  $\int_0^1 x \log(1+x) dx = \frac{1}{4}$
690.  $\int_0^1 x^m (\log x)^n dx = \frac{(-1)^n n!}{(m+1)^{n+1}}, \quad m > -1, n = 0, 1, 2, \dots$   
 If  $n \neq 0, 1, 2, \dots$  replace  $n!$  by  $\Gamma(n+1)$ .
691.  $\int_0^1 \frac{\log x}{1+x} dx = -\frac{\pi^2}{12}$
692.  $\int_0^1 \frac{\log x}{1-x} dx = -\frac{\pi^2}{6}$
693.  $\int_0^1 \frac{\log(1+x)}{x} dx = \frac{\pi^2}{12}$
694.  $\int_0^1 \frac{\log(1-x)}{x} dx = -\frac{\pi^2}{6}$
695.  $\int_0^1 (\log x)[\log(1+x)] dx = 2 - 2 \log 2 - \frac{\pi^2}{12}$
696.  $\int_0^1 (\log x)[\log(1-x)] dx = 2 - \frac{\pi^2}{6}$
697.  $\int_0^1 \frac{\log x}{1-x^2} dx = -\frac{\pi^2}{8}$
698.  $\int_0^1 \log\left(\frac{1+x}{1-x}\right) \cdot \frac{dx}{x} = \frac{\pi^2}{4}$
699.  $\int_0^1 \frac{\log x dx}{\sqrt{1-x^2}} = -\frac{\pi}{2} \log 2$
700.  $\int_0^1 x^m \left[\log\left(\frac{1}{x}\right)\right]^n dx = \frac{\Gamma(n+1)}{(m+1)^{n+1}}, \quad \text{if } m+1 > 0, n+1 > 0$
701.  $\int_0^1 \frac{(x^p - x^q) dx}{\log x} = \log\left(\frac{p+1}{q+1}\right), \quad (p+1 > 0, q+1 > 0)$
702.  $\int_0^1 \frac{dx}{\sqrt{\log\left(\frac{1}{x}\right)}} = \sqrt{\pi}, \text{ (same as integral 686)}$
703.  $\int_0^\infty \log\left(\frac{e^x + 1}{e^x - 1}\right) dx = \frac{\pi^2}{4}$
704.  $\int_0^{\pi/2} (\log \sin x) dx = \int_0^{\pi/2} \log \cos x dx = -\frac{\pi}{2} \log 2$
705.  $\int_0^{\pi/2} (\log \sec x) dx = \int_0^{\pi/2} \log \csc x dx = \frac{\pi}{2} \log 2$
706.  $\int_0^\pi x(\log \sin x) dx = -\frac{\pi^2}{2} \log 2$
707.  $\int_0^{\pi/2} (\sin x)(\log \sin x) dx = \log 2 - 1$

**DEFINITE INTEGRALS (Continued)**

$$708. \int_0^{\pi/2} (\log \tan x) dx = 0$$

$$709. \int_0^{\pi} \log(a \pm b \cos x) dx = \pi \log\left(\frac{a + \sqrt{a^2 - b^2}}{2}\right), \quad (a \geq b)$$

$$710. \int_0^{\pi} \log(a^2 - 2ab \cos x + b^2) dx = \begin{cases} 2\pi \log a, & a \geq b > 0 \\ 2\pi \log b, & b \geq a > 0 \end{cases}$$

$$711. \int_0^{\infty} \frac{\sin ax}{\sinh bx} dx = \frac{\pi}{2b} \tanh \frac{a\pi}{2b}$$

$$712. \int_0^{\infty} \frac{\cos ax}{\cosh bx} dx = \frac{\pi}{2b} \operatorname{sech} \frac{a\pi}{2b}$$

$$713. \int_0^{\infty} \frac{dx}{\cosh ax} = \frac{\pi}{2a}$$

$$714. \int_0^{\infty} \frac{x dx}{\sinh ax} = \frac{\pi^2}{4a^2}$$

$$715. \int_0^{\infty} e^{-ax} (\cosh bx) dx = \frac{a}{a^2 - b^2}, \quad (0 \leq |b| < a)$$

$$716. \int_0^{\infty} e^{-ax} (\sinh bx) dx = \frac{b}{a^2 - b^2}, \quad (0 \leq |b| < a)$$

$$717. \int_0^{\infty} \frac{\sinh ax}{e^{bx} + 1} dx = \frac{\pi}{2b} \operatorname{csc} \frac{a\pi}{b} - \frac{1}{2a}$$

$$718. \int_0^{\infty} \frac{\sinh ax}{e^{bx} - 1} dx = \frac{1}{2a} - \frac{\pi}{2b} \cot \frac{a\pi}{b}$$

$$719. \int_0^{\pi/2} \frac{dx}{\sqrt{1 - k^2 \sin^2 x}} = \frac{\pi}{2} \left[ 1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 k^4 + \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2 k^6 + \dots \right], \quad \text{if } k^2 < 1$$

$$720. \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 x} dx = \frac{\pi}{2} \left[ 1 - \left(\frac{1}{2}\right)^2 k^2 - \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 \frac{k^4}{3} - \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2 \frac{k^6}{5} - \dots \right], \quad \text{if } k^2 < 1$$

$$721. \int_0^{\infty} e^{-x} \log x dx = -\gamma = -0.5772157 \dots$$

$$722. \int_0^{\infty} e^{-x^2} \log x dx = -\frac{\sqrt{\pi}}{4} (\gamma + 2 \log 2)$$

$$723. \int_0^{\infty} \left( \frac{1}{1 - e^{-x}} - \frac{1}{x} \right) e^{-x} dx = \gamma = 0.5772157 \dots \quad [\text{Euler's Constant}]$$

$$724. \int_0^{\infty} \frac{1}{x} \left( \frac{1}{1+x} - e^{-x} \right) dx = \gamma = 0.5772157 \dots$$

For  $n$  even:

$$725. \int \cos^n x dx = \frac{1}{2^{n-1}} \sum_{k=0}^{n/2-1} \binom{n}{k} \frac{\sin(n-2k)x}{(n-2k)} + \frac{1}{2^n} \binom{n}{n/2} x$$

**DEFINITE INTEGRALS (Continued)**

$$726. \int \sin^n x \, dx = \frac{1}{2^{n-1}} \sum_{k=0}^{n/2-1} \binom{n}{k} \frac{\sin \left[ (n-2k) \left( \frac{\pi}{2} - x \right) \right]}{2k-n} + \frac{1}{2^n} \binom{n}{n/2} x$$

For  $n$  odd:

$$727. \int \cos^n x \, dx = \frac{1}{2^{n-1}} \sum_{k=0}^{(n-1)/2} \binom{n}{k} \frac{\sin(n-2k)x}{n-2k}$$

$$728. \int \sin^n x \, dx = \frac{1}{2^{n-1}} \sum_{k=0}^{(n-1)/2} \binom{n}{k} \frac{\sin \left[ (n-2k) \left( \frac{\pi}{2} - x \right) \right]}{2k-n}$$