HOMEWORK 2 SOLUTIONS

E-2) (a) The RMS speed is given by $v_{RMS} = \left[3 \frac{kT}{m}\right]^{\frac{1}{2}}$ where m is the mass of the atom. Since the atomic weight is 4.0026, 1 mole has a mass of 4.0026 grams $\Rightarrow$

$$m = (4.0026g)/N_A$$

$$v_{RMS} = \left[(3)\frac{(1.38\times10^{-23} J/K)(300 K)(6.02\times10^{23})}{4.0026\times10^{-3} kg}\right]^{\frac{1}{2}} \Rightarrow \boxed{v_{RMS} = 1367 m/s}$$

(b) The average speed is

$$\bar{v} = \left[\left(\frac{8}{\pi}\right)\frac{kT}{m}\right]^{1/2} \Rightarrow \boxed{\bar{v} = 1259 m/s}$$

(c) The most probable speed is

$$v_m = \left[2\frac{kT}{m}\right]^{\frac{1}{2}} \Rightarrow \boxed{v_m = 1116 m/s}$$

E-3) We have $g(E) = C E^{1/2} e^{-E/kT} \Rightarrow$

$$\langle E \rangle = \int_0^\infty E\, g(E)\, dE = C\int_0^\infty E^{3/2} e^{-E/kT} dE$$

Integrate by parts with

$$u = E^{3/2} \qquad dv = e^{-E/kT}$$
$$du = \tfrac{3}{2}E^{1/2} \qquad v = -kT e^{-E/kT}$$

$\Rightarrow$

$$\langle E \rangle = C\left\{ E^{3/2}(-kT)e^{-E/kT}\Big/_0^\infty - \int_0^\infty \tfrac{3}{2}E^{1/2}(-kT)e^{-E/kT} dE\right\}$$

$$= C\left\{0 + \tfrac{3}{2}kT\int_0^\infty E^{1/2}e^{-E/kT}dE\right\}$$

$$= \tfrac{3}{2}kT\int_0^\infty g(E)e^{-E/kT}dE = \left(\tfrac{3}{2}kT\right)\cdot(1)$$

since $g(E)$ must be normalized.

So our result is $\boxed{\langle E \rangle = \tfrac{3}{2}kT}$

8-7) This is basically like problem E-3 except that we need to use the neutron mass, 1.675×10^{-27} kg.

$$\langle v\rangle = \bar{v} = \left[\left(\frac{8}{\pi}\right)\frac{kT}{m}\right] = \left[\frac{(8)(1.38\times10^{-23}\,\mathrm{J/K})(300\,\mathrm{K})}{\pi\cdot 1.675\times10^{-27}\,\mathrm{kg}}\right]^{\frac{1}{2}}$$

$$\boxed{\bar{v} = 2509\ \mathrm{m/s}}$$

$v_m = \left[2\frac{kT}{m}\right]^{\frac{1}{2}}$ gives $\boxed{v_m = 2223\ \mathrm{m/s}}$

See prob 8-9 below for the last part of the problem

8-9) The speed distribution has the form

$$f(v) = Av^2 e^{-mv^2/2kT}$$

where A is a constant. The most probable value of v is the value at which $f(v)$ is maximum:

$$\frac{df}{dv} = 0 = A\left\{2ve^{-mv^2/2kT} + v^2\left(-\frac{2mv}{2kT}\right)e^{-mv^2/2kT}\right\}$$

$\Rightarrow$

$$= Av\left[2 - \frac{mv^2}{kT}\right]e^{-mv^2/2kT} \qquad = 0$$

So we need $mv^2/kT = 2 \quad \Rightarrow \quad v^2 = 2kT/m$

$$\boxed{v_m = \left[\frac{2kT}{m}\right]^{\frac{1}{2}}}$$

8-12) The probability that a given atom has energy E_i is given by the Boltzmann formula

$$f_i = Cg_i e^{-E_i/kT}$$

The population ratio of state 2 to state 1 is then

$$n_2/n_1 = f_2/f_1 = \frac{g_2 C e^{-E_2/kT}}{g_1 C e^{-E_1/kT}} = \left(\frac{g_2}{g_1}\right) e^{-(E_2-E_1)/kT}$$

Here

$$E_2 - E_1 = 4\times10^{-3}\,eV$$

and

$$kT = (8.618\times10^{-5}\,eV/k)\cdot(300\,k) = 0.0259\,eV$$

$$n_2/n_1 = (3/1)\, e^{-.004/.0259} = \boxed{2.57}$$

8-15) We have $g(E) = C E^{1/2} e^{-E/kT}$. To find E_m we want

$$\frac{dg}{dE} = 0 = C\left[\frac{1}{2} E^{-1/2} e^{-E/kT} - \frac{1}{kT} E^{1/2} e^{-E/kT}\right]$$

$$= C\left[\frac{1}{2}\frac{1}{E^{1/2}} - \frac{1}{kT} E^{1/2}\right] e^{-E/kT}$$

So we need

$$\frac{1}{2}\frac{1}{E^{1/2}} = \frac{1}{kT} E^{1/2} \Rightarrow \boxed{E_m = \frac{kT}{2}}$$

8-44) (a) We need the sum of the probabilities to be 1. In this problem there are only two states

$$f_1 = Ce^{-E_1/kT} = Ce^0 = C \qquad f_2 = Ce^{-\epsilon/kT}$$

$\Rightarrow$

$$C + Ce^{-\epsilon/kT} = C\left[1 + e^{-\epsilon/kT}\right] = 1 \qquad \boxed{C = \frac{1}{1+e^{-\epsilon/kT}}}$$

(b) The average energy is

$$\langle E\rangle = \sum_i E_i f_i$$

$\Rightarrow$

$$\langle E\rangle = Ce^0\cdot 0 + Ce^{-\epsilon/kT}\cdot\epsilon = \boxed{\epsilon\,\frac{e^{-\epsilon/kT}}{1+e^{-\epsilon/kT}}}$$

For $\underline{T\to 0}$

$$e^{-\epsilon/kT} \to e^{-\epsilon/0} \to e^{-\infty} = 0 \Rightarrow \boxed{\langle E\rangle = \epsilon\cdot\frac{0}{1+0} = 0}$$

For $T \to \infty$

$$e^{-\epsilon/kT} \to e^{-0} = 1$$

so

$$\langle E \rangle = \epsilon \frac{1}{1+1} \Rightarrow \boxed{\langle E \rangle = \epsilon/2}$$

(c) If we have N atoms then $U = N\langle E \rangle = N\epsilon \dfrac{e^{-\epsilon/kT}}{1+e^{-\epsilon/kT}}$

which we can write as

$$U = N\epsilon \left(1+e^{-\epsilon/kT}\right)^{-1} e^{-\epsilon/kT}$$

Then

$$C_v = \frac{dU}{dT} = N\epsilon \left\{ \left(1+e^{-\epsilon/kT}\right)^{-1} \frac{d}{dT} e^{-\epsilon/kT} + (-1)\left(1+e^{-\epsilon/kT}\right)^{-2} \times \left(\frac{d}{dT} e^{-\epsilon/kT}\right) \cdot e^{-\epsilon/kT} \right\}$$

$$= N\epsilon \left\{ \frac{1}{1+e^{-\epsilon/kT}} - \frac{e^{-\epsilon/kT}}{\left(1+e^{-\epsilon/kT}\right)^2} \right\} \frac{d}{dT} e^{-\epsilon/kT}$$

$$= N\epsilon \left\{ \frac{1+e^{-\epsilon/kT} - e^{-\epsilon/kT}}{\left(1+e^{-\epsilon/kT}\right)^2} \right\} e^{-\epsilon/kT} \left(-\frac{\epsilon}{k}\right)\left(-\frac{1}{T^2}\right)$$

$$\boxed{C_v = Nk \left(\frac{\epsilon}{kT}\right)^2 \frac{e^{-\epsilon/kT}}{\left(1+e^{-\epsilon/kT}\right)^2}}$$

(d)