# HOMEWORK 6 SOLUTIONS

6-3) $\psi(x) = Ae^{-x^2/2L^2}$

(a)
$$\frac{d^2}{dx^2}\psi(x) = \frac{d}{dx}A\left[-\frac{x}{L^2}e^{-x^2/2L^2}\right]$$

$$= A\left[-\frac{1}{L^2}e^{-x^2/2L^2} + \frac{x^2}{L^4}e^{-x^2/2L^2}\right] = \left(-\frac{1}{L^2}+\frac{x^2}{L^4}\right)Ae^{-x^2/2L^2}$$

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x) = E\,\psi(x)$$

$\Rightarrow$
$$V(x)\psi(x) = \frac{\hbar^2}{2m}\left[\cancel{-\frac{1}{L^2}} + \frac{x^2}{L^4}\right]Ae^{-x^2/2L^2} + \cancel{\frac{\hbar^2}{2mL^2}Ae^{-x^2/2L^2}}$$

$$= \frac{\hbar^2}{2mL^4}x^2Ae^{-x^2/2L^2} = \frac{1}{2}\frac{\hbar^2}{mL^4}x^2\,\psi(x)$$

so
$$\boxed{V(x) = \frac{1}{2}\frac{\hbar^2}{mL^4}x^2}$$

(b) This is the harmonic oscillator potential

6-5) (a) For $\Psi(x) = A\sin(kx-\omega t)$

$$\frac{\partial^2}{\partial x^2}\Psi(x) = -k^2A\sin(kx-\omega t)$$

$$\frac{\partial}{\partial t}\Psi(x) = -\omega A\cos(kx-\omega t)$$

We need
$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi + V\Psi = i\hbar\frac{\partial}{\partial t}\Psi$$

I think we are supposed to assume that the particle is free $\Rightarrow V(x) = 0$. Then we need

$$\frac{\hbar^2k^2}{2m}A\sin(kx-\omega t) = -i\hbar\omega A\cos(kx-\omega t)$$

which will not hold for all $x$ and $t$.

(b) $\Psi = A\left[\cos(kx-\omega t) + i\sin(kx-\omega t)\right]$

$$\frac{\partial^2}{\partial x^2}\Psi = -k^2 A\left[\cos(kx-\omega t) + i\sin(kx-\omega t)\right]$$

$$\frac{\partial}{\partial t}\Psi = A\left[+\omega\sin(kx-\omega t) - i\omega\cos(kx-\omega t)\right]$$

So we need

$$\frac{\hbar^2 k^2}{2m} A\left[\cos(kx-\omega t + i\sin(kx-\omega t)\right]$$

$$= \hbar\omega A\left[i\sin(kx-\omega t) + \cos(kx-\omega t)\right]$$

which holds for all x and t if $\frac{\hbar^2 k^2}{2m} = \hbar\omega$.

6-11) The energy is given by $E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$ so

$$n^2 = 2mE_nL^2/\pi^2\hbar^2 \Rightarrow n = \left[2mE_n\right]^{1/2}\frac{L}{\pi\hbar}.$$

Here

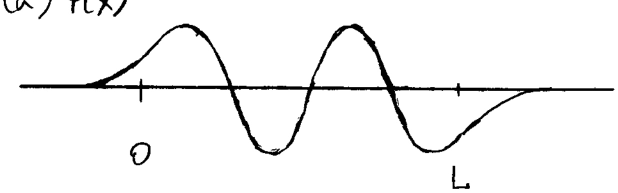
$$2mE_n = 2m\left(\tfrac{1}{2}mv^2\right) = (mv)^2$$

So

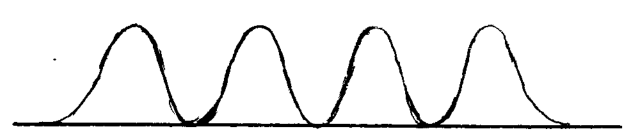
$$n = \frac{mvL}{\pi\hbar} = (10^{-9}\,\text{kg})(10^{-3}\,\text{m/s})(10^{-2}\,\text{m})/\pi(1.055\times10^{-34}\,\text{J}\cdot\text{s})$$

$$\boxed{n = 3.02\times10^{19}}$$

6-20) (a) $\psi(x)$




(b) $P(x)$



6-33) (a) The wave function is of the form

$$\psi(x) = C\,x\,e^{-\alpha x^2} \qquad \text{where} \qquad \alpha = \frac{\sqrt{km}}{2\hbar}$$

First normalize. We need

$$\int_{-\infty}^{\infty} \psi^*\psi\,dx = 1 = C^2 \int_{-\infty}^{\infty} x^2 e^{-2\alpha x^2}\,dx$$

$$= C^2 (2) \frac{\sqrt{\pi}}{4} (2\alpha)^{-3/2}$$

So

$$\underline{C^2 = \frac{2}{\sqrt{\pi}} (2\alpha)^{3/2}} \qquad \boxed{C = \left[\frac{2}{\sqrt{\pi}} (2\alpha)^{3/2}\right]^{\frac{1}{2}}}$$

(b) $$\langle x \rangle = \int_{-\infty}^{\infty} \psi^*\,x\,\psi\,dx$$

$$= C^2 \int_{-\infty}^{\infty} x^3 e^{-2\alpha x^2}\,dx = 0 \qquad \boxed{\langle x \rangle = 0}$$

since the integrand is an odd function

(c) $$\langle x^2 \rangle = \int_{-\infty}^{\infty} \psi^*\,x^2\,\psi\,dx = C^2 \int_{-\infty}^{\infty} x^4 e^{-2\alpha x^2}\,dx$$

$$= \left[\frac{2}{\cancel{\sqrt{\pi}}}(2\alpha)^{3/2}\right]\cdot\left[2\cdot\frac{3}{8}\cancel{\sqrt{\pi}}\,(2\alpha)^{-5/2}\right] = \frac{3}{2}\,\frac{1}{2\alpha} = \boxed{\frac{3}{4\alpha} = \langle x^2 \rangle}$$

OR

$$\langle x^2 \rangle = \frac{3}{4}\,\frac{2\hbar}{\sqrt{km}} = \frac{3}{2}\,\frac{\hbar}{\sqrt{km}}$$

(d) $$\langle V(x) \rangle = \left\langle \frac{1}{2}kx^2 \right\rangle = \frac{1}{2}k\,\langle x^2 \rangle$$

$$= \frac{1}{2}k\cdot\frac{3}{2}\,\frac{\hbar}{\sqrt{km}} = \frac{3}{4}\hbar\sqrt{\frac{k}{m}} \qquad \boxed{\langle V(x) \rangle = \frac{3}{4}\hbar\omega_0}$$

E-8) (a) For $\langle p \rangle$ we need $\frac{\hbar}{i}\frac{d}{dx}\psi = \frac{\hbar}{i}\frac{d}{dx}\,C\,x\,e^{-\alpha x^2}$

$$= \frac{\hbar}{i} C \left[e^{-\alpha x^2} - 2\alpha x^2 e^{-\alpha x^2}\right]$$

$$\langle p\rangle = \int_{-\infty}^{\infty} \psi^*(x)\,\frac{\hbar}{i}\frac{d}{dx}\,\psi(x)\,dx$$

$$= c^2\int_{-\infty}^{\infty} x e^{-\alpha x^2}\left(\frac{\hbar}{i}\right)\left[1-2\alpha x^2\right]e^{-\alpha x^2}\,dx$$

The integrand is an odd function of x so we get $\boxed{\langle p\rangle = 0}$

(b) Here we need $\left(\frac{\hbar}{i}\frac{d}{dx}\right)^2\psi = -\hbar^2\frac{d^2}{dx^2}\psi$

$$= -\hbar^2 c\,\frac{d}{dx}\left[1-2\alpha x^2\right]e^{-\alpha x^2}$$

$$= -\hbar^2 c\left[-4\alpha x + (1-2\alpha x^2)(-2\alpha x)\right]e^{-\alpha x^2}$$

$$= -\hbar^2 c\left[-6\alpha x + 4\alpha^2 x^3\right]e^{-\alpha x^2}$$

$$\langle p^2\rangle = \int_{-\infty}^{\infty}\psi^*\left(\frac{\hbar}{i}\frac{d}{dx}\right)^2\psi\,dx$$

$$= c^2(-\hbar^2)\int_{-\infty}^{\infty}\left(-6\alpha x^2 + 4\alpha^2 x^4\right)e^{-2\alpha x^2}\,dx$$

$$= \frac{2}{\cancel{\sqrt{\pi}}}(2\alpha)^{3/2}(-\hbar^2)(2)\left[-6\alpha\,\frac{\cancel{\sqrt{\pi}}}{4}\left(\frac{1}{2\alpha}\right)^{3/2} + 4\alpha^2\,\frac{3\cancel{\sqrt{\pi}}}{8}\left(\frac{1}{2\alpha}\right)^{5/2}\right]$$

$$= 4\hbar^2\left[\frac{6\alpha}{4} - \frac{12}{8}\,\frac{\alpha^2}{2\alpha}\right] = 4\hbar^2\left[\frac{3}{4}\alpha\right] \Rightarrow \boxed{\langle p^2\rangle = 3\alpha\hbar^2}$$

$$\langle p^2\rangle = 3\,\frac{\sqrt{km}}{2\hbar}\cdot\hbar^2 = \frac{3}{2}\sqrt{km}\,\hbar$$

(c) $KE = p^2/2m$

So

$$\langle KE\rangle = \frac{1}{2m}\langle p^2\rangle = \frac{3}{4m}\sqrt{km}\cdot\hbar \qquad \boxed{\langle KE\rangle = \frac{3}{4}\hbar\omega_0}$$

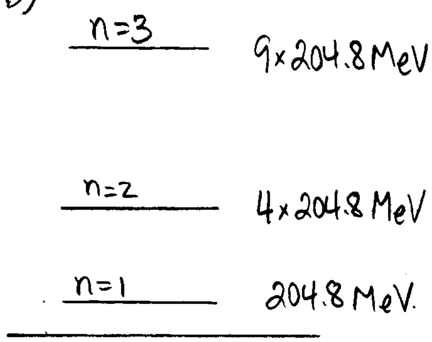
6-47) (a) The ground state energy is $E = \frac{\pi^2\hbar^2}{2mL^2} = \frac{(2\pi\hbar c)^2}{8\,mc^2L^2}$

Here $mc^2 = 938.3$ MeV, $2\pi\hbar c = hc = 1240\,\text{eV}\cdot\text{nm} = 1240\,\text{MeV}\cdot\text{fm}$.

So

$$E = \frac{(1240\,\text{MeV}\cdot\text{fm})^2}{8\,(938.3\,\text{MeV})(1\,\text{fm})^2} = 204.8\ \text{MeV}$$

(b)




(c) $E = hf = \frac{hc}{\lambda} \Rightarrow \lambda = \frac{hc}{E}$

$n=2 \rightarrow n=1 \Rightarrow E = 3\times 204.8\,\text{MeV}$

$$\lambda = \frac{1240\ \text{MeV}\cdot\text{fm}}{3\times 204.8\,\text{MeV}} \qquad \boxed{\lambda = 2.02\ \text{fm}}$$

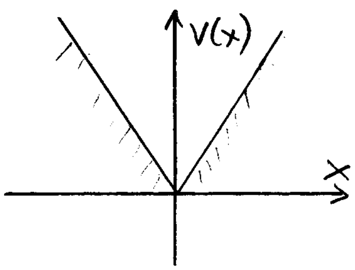
(d) $n=3 \rightarrow n=2 \Rightarrow E = 5\times 204.8\,\text{MeV}$

$\boxed{\lambda = 1.21\ \text{fm}}$

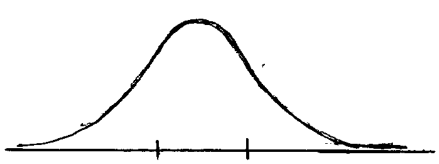
(e) $n=3 \rightarrow n=1 \Rightarrow E = 8\times 204.8\ \text{MeV}$

$\boxed{\lambda = 0.76\ \text{fm}}$

6-59)




Ground State (zero nodes)



First Ex State (one node @ center)