HOMEWORK 7 SOLUTIONS

E-9)

$$\Psi = \left[\frac{\alpha}{2\pi}\right]^{1/4}\left[e^{-i\omega_0 t} + 2\sqrt{\alpha}\,x\,e^{-3i\omega_0 t}\right]e^{-\alpha x^2}$$

$$\Psi^* = \left[\frac{\alpha}{2\pi}\right]^{1/4}\left[e^{+i\omega_0 t} + 2\sqrt{\alpha}\,x\,e^{+3i\omega_0 t}\right]e^{-\alpha x^2}$$

$\Rightarrow$

$$\langle x\rangle = \int_{-\infty}^{\infty}\Psi^*\,x\,\Psi\,dx$$

$$= \left(\frac{\alpha}{2\pi}\right)^{\frac{1}{2}}\int_{-\infty}^{\infty}\left[e^{i\omega_0 t} + 2\sqrt{\alpha}\,x\,e^{3i\omega_0 t}\right]x\left[e^{-i\omega_0 t} + 2\sqrt{\alpha}\,x\,e^{-3i\omega_0 t}\right]e^{-2\alpha x^2}\,dx$$

$$= \left(\frac{\alpha}{2\pi}\right)^{\frac{1}{2}}\int_{-\infty}^{\infty}\left[x + 2\sqrt{\alpha}\,x^2 e^{2i\omega_0 t} + 2\sqrt{\alpha}\,x^2 e^{-2i\omega_0 t} + 4\alpha x^3\right]e^{-2\alpha x^2}\,dx$$

But $\int_{-\infty}^{\infty} x e^{-2\alpha x^2}\,dx$ and $\int_{-\infty}^{\infty} x^3 e^{-2\alpha x^2}\,dx$ are both zero, so we have

$$\langle x\rangle = \left(\frac{\alpha}{2\pi}\right)^{1/2} 2\sqrt{\alpha}\left(e^{2i\omega_0 t} + e^{-2i\omega_0 t}\right)\int_{-\infty}^{\infty} x^2 e^{-2\alpha x^2}\,dx$$

$$= \left(\frac{\alpha}{2\cancel{\pi}}\right)^{1/2}\cancel{2}\sqrt{\alpha}\left(2\cos 2\omega_0 t\right)\cancel{(2)}\left(\frac{1}{\cancel{4}}\right)\sqrt{\cancel{\pi}}\,(2\alpha)^{-3/2}$$

$$\boxed{\langle x\rangle = \frac{1}{2\sqrt{\alpha}}\cos 2\omega_0 t}$$

E-10) (a) In the ground state the energy is $E = \frac{1}{2}\hbar\omega_0 = \frac{1}{2}\hbar\sqrt{\frac{k}{m}}$ and $V(x) = \frac{1}{2}kx^2$. The classically allowed region is the region where $V(x) < E$ so the limits are the points where $E = V(x)$

$\Rightarrow$

$$\cancel{\tfrac{1}{2}}\hbar\sqrt{\frac{k}{m}} = \cancel{\tfrac{1}{2}}kx^2$$

$\Rightarrow$

$$x^2 = \frac{\hbar}{\sqrt{km}} \qquad \boxed{x = \pm\left[\frac{\hbar}{\sqrt{km}}\right]^{\frac{1}{2}}} = \pm x_0$$

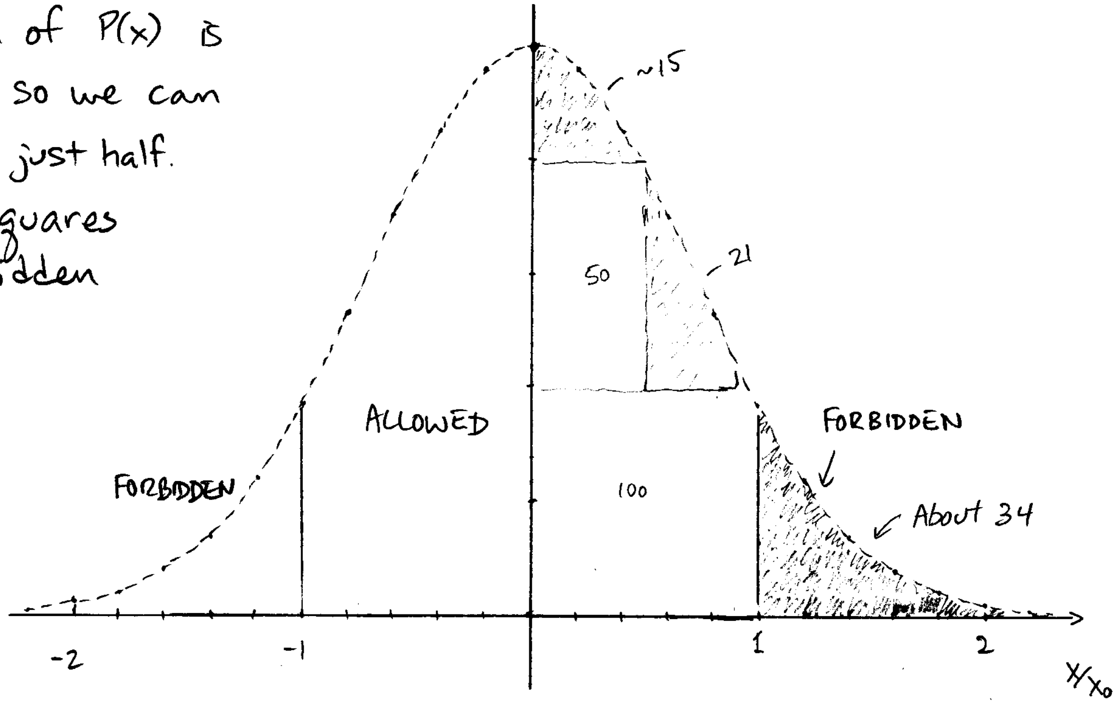
But

$$\alpha = \frac{\sqrt{km}}{2\hbar} \Rightarrow \frac{\sqrt{km}}{\hbar} = 2\alpha \Rightarrow \underline{x_0 = \frac{1}{\sqrt{2\alpha}}}$$

(b) The wave function goes like $\psi(x) = N e^{-\alpha x^2}$, so

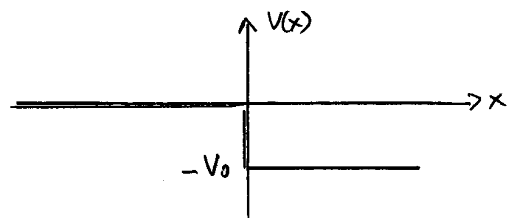
$P(x) = |\psi|^2 \propto e^{-2\alpha x^2} = e^{-x^2/x_0^2}$

The graph of $P(x)$ is symmetric so we can work with just half. I get 34 squares for the forbidden region and 186 for the allowed region. So my result is



P = probability in the forbidden region $= \dfrac{34}{186+34} = \dfrac{34}{220} = \boxed{0.155}$

6-38) (a) We have $E = 2V_0$, and for $x<0$ the wave number is k_1. Here $V(x)=0$ so



$$k_1 = \left[\frac{2mE}{\hbar^2}\right]^{\frac{1}{2}} = \left[\frac{4mV_0}{\hbar^2}\right]^{\frac{1}{2}}$$

For $x>0$ $V(x) = -V_0$ so $-\dfrac{\hbar^2}{2m}\dfrac{d^2}{dx^2}\psi = (E+V_0)\psi = 3V_0\psi$

$\Rightarrow$

$$\boxed{k_2 = \left[\frac{6mV_0}{\hbar^2}\right]^{\frac{1}{2}} = \sqrt{1.5}\, k_1}$$

(b) $R = \left(\dfrac{k_1-k_2}{k_1+k_2}\right)^2 = \left(\dfrac{1-\sqrt{1.5}}{1+\sqrt{1.5}}\right)^2$ $\boxed{R = 0.010}$

(c) $\boxed{T = 1-R = 0.99}$

(d) $N = T \times N_0 = (0.99)\times 10^6 = \boxed{9.9\times 10^5}$ Classically all go to $+x$.

6-41) We have $A+B=C$ (1) and $k_1A-k_1B=k_2C$ (2)

$\Rightarrow$

$$A-B=\frac{k_2}{k_1}C \quad (3)$$

Adding (1) and (3)

$$2A=\left(1+\frac{k_2}{k_1}\right)C \qquad C=\left(\frac{2}{1+k_2/k_1}\right)A$$

$$\Rightarrow \quad \boxed{C=\left(\frac{2k_1}{k_1+k_2}\right)A}$$

From (1)

$$B=C-A=\frac{2k_1}{k_1+k_2}A-A=\frac{2k_1-(k_1+k_2)}{k_1+k_2}A \qquad \boxed{B=\frac{k_1-k_2}{k_1+k_2}A}$$

E-11) We will use $T=16\frac{E}{V_0}\left(1-\frac{E}{V_0}\right)e^{-2\alpha a}$ where $\alpha=\left[\frac{2m(V_0-E)}{\hbar^2}\right]^{\frac{1}{2}}$

(a) $$\alpha a=\frac{[2(5.11\times10^5 eV)(0.3eV)]^{\frac{1}{2}}}{(1240\,eV\cdot nm/2\pi)}\cdot 10nm = 28.06$$

$$T=16\left(\frac{3}{3.3}\right)\left(1-\frac{3}{3.3}\right)e^{-2(28.06)} = \boxed{5.5\times10^{-25}}$$

(b)

$$\alpha a=\frac{[2(5.11\times10^5 eV)(2eV)]^{\frac{1}{2}}}{1240eV\cdot nm/2\pi}\cdot 10\,nm = 72.46 \Rightarrow \boxed{T=4.4\times10^{-63}}$$

(c) $\alpha a = 7.246$ $\quad \boxed{T=1.95\times10^{-6}}$

(d) $$\alpha a=\left(\frac{[2(938\times10^6 eV)(2eV)]^{\frac{1}{2}}}{1240/2\pi\ eV\cdot nm}\right)\cdot(0.1nm) = 31.05$$

$$\boxed{T=4.1\times10^{-27}}$$

7-5) We have $L_2=2L_1$ and $L_3=4L_1$. The wave function is of the form

$$\psi = A\sin k_1x\,\sin k_2y\,\sin k_3z$$

where k_1, k_2 and k_3 need to be chosen to make $\psi\to0$ at

$x = L_1$, $y = L_2$ and $z = L_3$. So we need $k_1 L_1 = n_1\pi \Rightarrow k_1 = \frac{n_1\pi}{L_1}$

$k_2 L_2 = n_2\pi \Rightarrow k_2 = n_2\pi/L_2$ and $k_3 L_3 = n_3\pi \Rightarrow k_3 = n_3\pi/L_3$

Then

$$E = \frac{\hbar^2}{2m}\left[k_1^2 + k_2^2 + k_3^2\right] = \frac{\hbar^2}{2m}\left[\left(\frac{n_1\pi}{L_1}\right)^2 + \left(\frac{n_2\pi}{L_2}\right)^2 + \left(\frac{n_3\pi}{L_3}\right)^2\right]$$

$$\Rightarrow \quad E = \frac{\hbar^2\pi^2}{2mL_1^2}\left[n_1^2 + \left(\frac{n_2}{2}\right)^2 + \left(\frac{n_3}{4}\right)^2\right] = \underline{\frac{\hbar^2\pi^2}{2mL_1^2}\left(\frac{1}{16}\right)\left[16n_1^2 + 4n_2^2 + n_3^2\right]}$$

$$E = C_0\left[16n_1^2 + 4n_2^2 + n_3^2\right]$$

	n_1	n_2	n_3	
1)	1	1	1	$E_{111} = 21\,C_0$
2)	1	1	2	$E_{112} = 24\,C_0$
3)	1	1	3	$E_{113} = 29\,C_0$
4)	1	2	1	$E_{121} = 33\,C_0$
5)	1	2	2	$E_{122} = 36\,C_0$ } Degenerate
6)	1	1	4	$E_{114} = 36\,C_0$ }
7)	1	2	3	$E_{123} = 41\,C_0$
8)	1	1	5	$E_{115} = 45\,C_0$
9)	1	2	4	$E_{124} = 48\,C_0$
10)	1	3	1	$E_{131} = 53\,C_0$

E-12)

E	Degen.	Quantum #'s
$\frac{9}{2}\hbar\omega_0$	10	003, 030, 300, 012, 021, 102, 201, 120, 210, 111
$\frac{7}{2}\hbar\omega_0$	6	002, 020, 200, 110, 101, 011
$\frac{5}{2}\hbar\omega_0$	3	001, 010, 101
$\frac{3}{2}\hbar\omega_0$	1	000