HOMEWORK 8 SOLUTIONS

7-9) (a) $l = 0, 1, 2$

(b) $l = 0 \qquad m = 0$

$l = 1 \qquad m = 0, \pm 1$

$l = 2 \qquad m = 0, \pm 1, \pm 2$

(c) $l = 0 \Rightarrow$ 1 choice for m, $m_s = \pm \frac{1}{2} \Rightarrow$ 2 states

$l = 1$ 3 choices for m, $m_s = \pm \frac{1}{2} \Rightarrow 3 \times 2 = 6$ states

$l = 2$ 5 " " ", " $\Rightarrow 5 \times 2 = 10$ states.

$\boxed{\text{TOTAL } 18}$

7-17) (a) In the 6f state $\boxed{n = 6,\ l = 3}$

(b) $E = -13.6\,\text{eV}/n^2 = -13.6\,\text{eV}/36 = \boxed{-0.378\,\text{eV}}$

(c) $L^2 = l(l+1)\hbar^2 = (3)(4)\hbar^2$

So

$$L = \sqrt{12}\,\hbar = \sqrt{12}\,(1.055 \times 10^{-34}\,\text{J}\cdot\text{s}) = \boxed{3.65 \times 10^{-34}\,\text{J}\cdot\text{s}}$$

(d) $L_z = m\hbar \qquad m = 0, \pm 1, \pm 2, \pm 3$

$\Rightarrow$

$$\boxed{L_z = 0,\ \pm 1.055 \times 10^{-34}\,\text{J}\cdot\text{s},\ \pm 2.110 \times 10^{-34}\,\text{J}\cdot\text{s},\ \pm 3.165 \times 10^{-34}\,\text{J}\cdot\text{s}}$$

E-13) $R(r) = Cre^{-r/2a_0}$, so the radial probability distribution is

$$P(r) = r^2 |R(r)|^2 = C^2 r^4 e^{-r/a_0}$$

(a) First normalize. We need $\int_0^\infty P(r)\,dr = 1$

$x \equiv r/a_0$

$dr = a_0\,dx$

$$\int_0^\infty C^2 r^4 e^{-r/a_0}\,dr = C^2 a_0^5 \int_0^\infty x^4 e^{-x}\,dx = C^2 a_0^5 \cdot 4!$$

$$\Rightarrow \underline{C^2 = 1/a_0^5 \cdot 4!}$$

Now

$$\langle r \rangle = \int_0^\infty r P(r)\,dr = \int_0^\infty c^2 r^5 e^{-r/a_0}\,dr$$

$$= c^2 a_0^6 \int_0^\infty x^5 e^{-x}\,dx = \left(\frac{1}{a_0^5 \cdot 4!}\right) a_0^6 \cdot 5! = 5a_0 \qquad \boxed{\langle r \rangle = 5a_0}$$

(b) To find the most probable r we want

$$\frac{d}{dr} P(r) = 0 = \frac{d}{dr} c^2 r^4 e^{-r/a_0} = c^2\left[4r^3 e^{-r/a_0} - \frac{1}{a_0} r^4 e^{-r/a_0}\right]$$

$$= c^2 r^3 e^{-r/a_0}\left[4 - r/a_0\right]$$

$\Rightarrow$

$$\frac{r}{a_0} = 4 \qquad \boxed{r_m = 4a_0}$$

E-14) We can find the normalized radial wave function for the ground state in Table 7-2.

$$R(r) = R_{10}(r) = 2a_0^{-3/2} e^{-r/a_0}$$

It follows that

$$P(r) = r^2 |R(r)|^2 = \frac{4}{a_0^3} r^2 e^{-2r/a_0}$$

(a) $\langle r \rangle = \int_0^\infty r P(r)\,dr$

$$x = \frac{2r}{a_0} \qquad dr = \frac{a_0}{2}\,dx$$

$$= \frac{4}{a_0^3}\int_0^\infty r^3 e^{-2r/a_0}\,dr$$

$$= \left(\frac{4}{a_0^3}\right)\int_0^\infty \left(\frac{a_0}{2}x\right)^3 e^{-x}\,\frac{a_0}{2}\,dx = \frac{4}{a_0^3}\,\frac{a_0^4}{16}\int_0^\infty x^3 e^{-x}\,dx$$

$$= \left(\frac{a_0}{4}\right)\cdot 3! = \boxed{\frac{3}{2}a_0}$$

(b) $V(r) = -\frac{e^2}{4\pi\epsilon_0}\frac{1}{r}$ so $\langle V \rangle = -\frac{e^2}{4\pi\epsilon_0}\cdot\left\langle \frac{1}{r}\right\rangle$

$$\left\langle \frac{1}{r}\right\rangle = \int_0^\infty \frac{1}{r} P(r)\,dr = \frac{4}{a_0^3}\int_0^\infty r e^{-2r/a_0}\,dr = \frac{4}{a_0^3}\int_0^\infty \left(\frac{a_0}{2}x\right) e^{-x}\left(\frac{a_0}{2}\right)dx$$

$$= \frac{4}{a_0^3}\,\frac{a_0^2}{4}\cdot 1! = 1/a_0$$

$$\boxed{\langle V\rangle = -\frac{e^2}{4\pi\epsilon_0}\cdot\frac{1}{a_0}}$$

(c) From Eq. (4-19) $\quad a_0 = \frac{\hbar^2}{mke^2} = \frac{4\pi\epsilon_0}{e^2}\frac{\hbar^2}{m}$

$\Rightarrow$

$$\langle V\rangle = -\left(\frac{e^2}{4\pi\epsilon_0}\right)\left(\frac{e^2}{4\pi\epsilon_0}\right)\frac{m}{\hbar^2} = -\left(\frac{e^2}{4\pi\epsilon_0}\frac{1}{\hbar c}\right)^2 mc^2 = -27.2\,eV.$$

For the ground state

$$E = -\frac{mc^2}{2}\left(\frac{e^2}{4\pi\epsilon_0}\cdot\frac{1}{\hbar c}\right)^2$$

So

$$\langle KE\rangle = E - \langle V\rangle = -\frac{mc^2}{2}\left(\frac{e^2}{4\pi\epsilon_0}\cdot\frac{1}{\hbar c}\right)^2 - (-mc^2)\left(\frac{e^2}{4\pi\epsilon_0}\cdot\frac{1}{\hbar c}\right)^2$$

$$\boxed{\langle KE\rangle = +\frac{mc^2}{2}\left(\frac{e^2}{4\pi\epsilon_0}\cdot\frac{1}{\hbar c}\right)^2 = +13.6\,eV}$$

E-15) We just need to integrate the probability distribution over the region of interest. From E-14 $\quad P(r) = (4/a_0^3)\, r^2 e^{-2r/a_0}$ $\quad$ so

$$P(r>a_0) = \int_{a_0}^{\infty} P(r)\,dr = \left(\frac{4}{a_0^3}\right)\int_{a_0}^{\infty} r^2 e^{-2r/a_0}\,dr.$$

As in E-14, define

$$x = 2r/a_0 \qquad r = \frac{a_0}{2}x$$

Then for $r = a_0$ we have $\quad x = 2r/a_0 = 2$

So

$$P(r>a_0) = \left(\frac{4}{a_0^3}\right)\int_2^{\infty}\left(\frac{a_0}{2}x\right)^2 e^{-x}\left(\frac{a_0}{2}\right)dx = \frac{1}{2}\int_2^{\infty} x^2 e^{-x}\,dx.$$

Integrate by parts. Let $u = x^2 \quad dv = e^{-x}dx \quad \Rightarrow \quad v = -e^{-x}\,dx$

$$\int x^2 e^{-x}dx = x^2(-e^{-x}) - \int 2x(-)e^{-x}dx = -x^2e^{-x} + 2\int xe^{-x}dx$$

Integrate by parts again with $u = x \quad dv = e^{-x}dx$

$$\int x^2e^{-x}dx = -x^2e^{-x} + 2(-xe^{-x}) - 2\int(-e^{-x})\,dx$$

$$= -x^2e^{-x} - 2xe^{-x} + 2\int e^{-x}dx = (-x^2 - 2x - 2)e^{-x}$$

So $$P(x>a_0) = \frac{1}{2}\left[-x^2-2x-2\right]e^{-x}\Big/_2^{\infty}$$

$$= 0 - \frac{1}{2}\left[-4-4-2\right]e^{-2} = \boxed{5e^{-2} = 0.677}$$

E-16) (a) The energy splitting is $\Delta E = \alpha^4 mc^2 / 2n^3 \ell(\ell+1)$

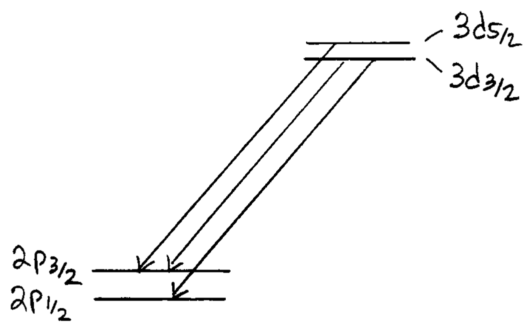
2p state: $n=2$ $\ell=1$

$$\Delta E = \left(\tfrac{1}{137}\right)^4 (5.11\times10^5\,eV) / (2)(2)^3(1)(2) = \boxed{4.53\times10^{-5}\,eV}$$

3d state: $n=3$ $\ell=2$

$$\Delta E = \left(\tfrac{1}{137}\right)^4 (5.11\times10^5\,eV) / 2(3)^3(2)(3) = \boxed{4.48\times10^{-6}\,eV}$$

(b)



7-33) n=2 Here we can have $\ell=0$ or $\ell=1$. For $\ell=0$ $j=\frac{1}{2}$ only.
For $\ell=1$, $j=\ell\pm\frac{1}{2} = \frac{1}{2}$ or $\frac{3}{2}$

$$\boxed{2s_{1/2},\ 2p_{1/2},\ 2p_{3/2}}$$

n=4 $\Rightarrow$ $\ell=0,1,2,3$. Again for $\ell=0$ we have $j=\frac{1}{2}$ only, and for all other ℓ values $j=\ell\pm\frac{1}{2}$

$$\boxed{4s_{1/2},\ 4p_{1/2},\ 4p_{3/2},\ 4d_{3/2},\ 4d_{5/2},\ 4f_{5/2},\ 4f_{7/2}}$$

7-35) (a) $j = \ell \pm \frac{1}{2}$ so we have $\boxed{j = 3/2 \text{ or } 5/2}$

(b) We have $J^2 = j(j+1)\hbar^2$ so $J = \sqrt{j(j+1)}\,\hbar$

For

$j = 3/2$ $\quad J = \sqrt{\frac{3}{2}\left(\frac{5}{2}\right)}\,\hbar = \boxed{1.936\hbar}$

$j = 5/2$ $\quad J = \sqrt{\left(\frac{5}{2}\right)\left(\frac{7}{2}\right)}\,\hbar = \boxed{2.958\hbar}$

(c) $J_z = m_j\hbar$ where $m_j = -j, -j+1, \cdots, +j$

So

$$\boxed{J_z = \pm\frac{1}{2}\hbar, \pm\frac{3}{2}\hbar, \pm\frac{5}{2}\hbar}$$