FINAL EXAM SOLUTIONS

1. (a) We can get $\vec{B}$ from $\vec{V} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$. Our electric field has only an $x$ component so

$$\nabla \times \vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & E_x \end{vmatrix} = \hat{y} \frac{2}{2} E_x = \hat{y} E_0 k \cos k z \ e^{-iwt} = -\frac{\partial \vec{B}}{\partial t}$$

Integrate over time to get $\vec{B}$:

$$\vec{B} = -\hat{y} E_0 k \cos k z \int e^{-iwt} \, dt = -\hat{y} E_0 k \cos k z \left(-\frac{1}{iw}\right) e^{-iwt}$$

$$\Rightarrow \boxed{\vec{B} = -i E_0 \frac{k}{w} \hat{y} \cos k z \ e^{-iwt}}$$

As usual, we will need $\omega k = c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$ which comes from

$$\nabla \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}.$$ 

(b) Let's take the real parts of $\vec{E} + \vec{B}$. $e^{-iwt} \cos \omega t$ gives

$$\boxed{\vec{E} = E_0 \hat{x} \sin k z \cos \omega t} \quad \boxed{\vec{B} = -\frac{E_0}{c} \hat{y} \cos k z \sin \omega t}$$

The energy per unit volume is

$$u_{em} = \frac{E_0^2}{2} E^2 + \frac{\mu_0}{2} B^2 = \frac{1}{2} \left[ E_0^2 \sin^2 k z \cos^2 \omega t + \frac{1}{\mu_0} (E_0^2 c^2) \cos^2 k z \sin^2 \omega t \right]$$

To get the total stored energy we integrate over all space. Using the fact that $\sin^2 k z$ and $\cos^2 k z$ both have average values of $\frac{1}{2}$ we have $\int \sin^2 k z \, dV = \int \cos^2 k z \, dV = \frac{1}{2} V$. Then with $\frac{1}{2} E^2 = \frac{1}{2} E_0^2 \mu_0$ we get

$$u = \frac{1}{2} \left[ E_0 E_0 \left( \frac{1}{2} V \right) \cos^2 \omega t + \frac{1}{\mu_0} (E_0 \mu_0) E_0^2 \left( \frac{1}{2} V \right) \sin^2 \omega t \right]$$

$$= \frac{1}{4} E_0 E_0^2 V (\cos^2 \omega t + \sin^2 \omega t) \Rightarrow \boxed{u = \frac{1}{4} E_0 E_0^2 V} \text{ (time indep.)}$$
2) We will use cylindrical coordinates with $\hat{z}$ out of the page. The magnetic field in the solenoid is uniform with 

$$\mathbf{B} = \mu_0 n I \hat{z} = \mu_0 (I_0 + \alpha t) \hat{z}$$

Since $\mathbf{B}$ is increasing there will be an induced EMF $E = -\frac{d\mathbf{B}}{dt}$. 

Think about a loop of radius $s$. The flux thru the loop is 

$$\Phi = \mathbf{B} \cdot \mathbf{A} = \mu_0 n I (\pi s^2) \Rightarrow \frac{d\Phi}{dt} = \mu_0 n \pi s^2 \frac{dI}{dt} = \mu_0 n \pi s^2 \alpha$$

$E$ = induced emf $= \oint E \cdot d\ell = (\Phi \text{ diff}) E_{\alpha}$ 

So 

$$E_{\alpha} = (\frac{1}{\text{diff}}) \frac{d\Phi}{dt} = (\frac{1}{\text{diff}}) \mu_0 n \pi s^2 \alpha = \mu_0 n \alpha \frac{s}{2}$$

Using right-hand rules, $E_{\alpha}$ should be negative. 

Notice that $\mathbf{E} \times \mathbf{B}$ is radially inward. Combining our results gives

$$S = \frac{1}{\mu_0} (-\mu_0 n \alpha \frac{s}{2} \hat{\theta}) \times (\mu_0 n I \hat{z}) = [-\mu_0 n^2 I \alpha \frac{s}{2} \hat{\theta}]$$

(b) Our result for $S$ says that there is energy flowing radially inward. This is consistent with the idea that there is energy stored in the magnetic field. $\mathbf{B}$ exists everywhere inside the solenoid, and $\mathbf{B}$ is increasing, so we need to have energy flow inward from the coil windings to the interior.

3) It's important to recognize that the charge density on the plates will be different for the two halves. Let's start by thinking about $E$. 

Since $d \ll a$ the electric field is uniform between the plates, and the voltage difference is 

$$\Delta V = \int E \cdot d\ell = E \cdot d \Rightarrow E = \frac{\Delta V}{d}$$

The electric field is the same in both halves of the capacitor. 

Next think about $D$. For the "empty" half $D = \varepsilon_0 E$, and for the dielectric half $D = \varepsilon_0 K E$. 

The last step is to relate \( \mathbf{D} \) to the free charge on the plates.

In general \( \nabla \cdot \mathbf{B} = \rho_f \), so by Gauss's Law

\[
\oint \mathbf{D} \cdot \mathbf{n} \, dA = \mathbf{D} \cdot \mathbf{a} = \rho_f, \text{enclosed} = \sigma_f \cdot \mathbf{a} \Rightarrow \sigma_f = \mathbf{D}
\]

Thus, for the empty side \( \mathbf{E} = \mathbf{D} = \varepsilon_0 \varepsilon_0 \mathbf{V}/d \) and for the dielectric side \( \mathbf{E} = \kappa \varepsilon_0 \varepsilon_0 \mathbf{V}/d \) and the total charge is

\[
Q = \left( \frac{a^2}{2} \right) \left( \frac{\varepsilon_0 \mathbf{V}}{d} \right) + \left( \frac{a^2}{2} \right) \left( \kappa \varepsilon_0 \mathbf{V}/d \right) \Rightarrow Q = \frac{a^2}{2} \left( \frac{\varepsilon_0 \mathbf{V}}{d} \right)
\]

4)(a) For a perfect conductor \( \mathbf{E} = 0 \), but \( \mathbf{F} = \sigma \mathbf{E} \) and \( \mathbf{F} \) can't be infinite so \( \mathbf{E} = 0 \). Then \( \nabla \times \mathbf{E} = 0 \) and so by Maxwell's Equations

\[
\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} = 0 \Rightarrow \mathbf{B} = \text{constant. for a perfect conductor.}
\]

(b) \( \nabla \cdot \mathbf{B} = 0 \Rightarrow \oint \mathbf{B} \cdot \mathbf{n} \, dA = 0 \)

\( \Rightarrow \) \( \mathbf{B}_z \cdot \mathbf{a} - \mathbf{B}_z \cdot \mathbf{a} = 0 \Rightarrow \mathbf{B}_z = \mathbf{B}_z \) (above) (below)

\( \Rightarrow \) \( \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \Rightarrow \oint \mathbf{B} \, d\mathbf{l} = \mu_0 I \text{encld.}
\)

\( \Rightarrow \) \( \mathbf{B}_y \cdot \mathbf{l} - \mathbf{B}_y \cdot \mathbf{l} = \mu_0 K \cdot \mathbf{l} \Rightarrow \mathbf{B}_y = \mathbf{B}_y \) (below) (above) \( \Rightarrow \mathbf{B}_y = \mathbf{B}_y = \mu_0 K \)

(c) \( \mathbf{B} = 0 \) below the surface \( \Rightarrow \) above \( \mathbf{B}_z \) must be zero but \( \mathbf{B}_y \) can be nonzero from any induced surface currents.

To cancel the \( \mathbf{E} \) field components we need to postulate a mirror wire, distance \( d \) below the surface, with \( \mathbf{I} \) into the page.

(d) We can get \( K \) from \( \mathbf{B}_y \). Just below the surface \( \mathbf{B}_y = 0 \).

Just above, \( \mathbf{B}_y \) can be gotten from the wire and it's mirror.

\( \mathbf{B} \) at a distance \( d \) is (by Ampere's Law) \( \mathbf{B} = \mu_0 \mathbf{I}/\pi d \). We get identical contributions from both wires \( \Rightarrow \mathbf{B}_y (\text{above}) = \mu_0 \mathbf{I}/\pi d \).
and then from the boundary conditions
\[ \mu_0 K = B_y^{\text{(below)}} - B_y^{\text{(above)}} = -\mu_0 I/\pi d \]

5) In empty space \( \nabla^2 V = 0 \). The general form of the solution in spherical coordinates is
\[ V = \sum_{l=0}^{\infty} \left( A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos \theta) \]

We need \( V \to 0 \) as \( r \to \infty \) so all the \( A_l \)'s are zero. Of the \( B_l \) terms, the largest one as \( r \to \infty \) is the one with the smallest \( l \).

By symmetry, the net change on our sphere must be zero \( \Rightarrow B_0 = 0 \). The sphere will have a dipole moment, and so we expect that as \( r \to \infty \)
\[ V \to \frac{B_1}{r^2} P_1(\cos \theta) = \frac{B_1 \cos \theta}{r^2} \Rightarrow \text{a typical dipole potential.} \]

We can find \( B_1 \) by matching at \( r=R \). In general
\[ V(R,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos \theta) = \begin{cases} +V_0 & 0 < \theta < \frac{\pi}{2} \\ -V_0 & \frac{\pi}{2} < \theta < \pi \end{cases} \]

To get \( B_1 \), multiply by \( P_1(\cos \theta) \) \( \sin \theta \) \( d\theta \) and integrate. By Fourier's trick
\[ \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} \int_0^\pi P_l(\cos \theta) P_1(\cos \theta) \sin \theta \, d\theta = \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} \left( \frac{2}{3} \right) \delta_{l1} = \frac{2}{3} \frac{B_1}{R^2} \]

\[ = \int_0^\pi V(R,\theta) \cos \theta \sin \theta \, d\theta = \left(3\right) \int_0^{\pi/2} V_0 \cos \theta \sin \theta \, d\theta = 2 \int_0^{\pi/2} V_0 \left( \frac{\sin^2 \theta}{2} \right) \, d\theta = V_0 \]

\[ V(R,\theta) \to \frac{3}{2} V_0 \frac{R^2}{r^2} \cos \theta \]

\[ B_1 = \frac{3}{2} V_0 R^2 \]

\[ V(r,\theta) \to \frac{3}{2} V_0 \frac{R^2}{r^2} \cos \theta \]