Target Space in Minimal String Theory

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Seiberg and D.S.	hep-th/0312170
Kutasov, Okuyama, Park, Seiberg and D.S.	hep-th/0406030
Maldacena, Moore, Seiberg and D.S.	hep-th/0408039

Motivation

Minimal string theories, a.k.a. "strings in $d < 2$," are simple and tractable toy models.

They are dual to certain random matrix models. This provides us with the simplest known example of open/closed duality and holography.

Open/closed duality is especially interesting here, because it relates two exactly solvable theories.

The matrix model also gives a precise nonperturbative definition to minimal string theory.

Thus, the minimal string is an ideal laboratory for studying nonperturbative effects in string theory.

We will use the matrix model to study the target space of the minimal string. We will see that nonperturbative effects are important.

Outline

- Review of minimal string theory
- Semiclassical target space – worldsheet description of FZZT branes
- Exact target space – matrix model description of FZZT branes
- General lessons and relation to other work

Review of minimal string theory

Minimal string theory has a traditional worldsheet construction.

Want worldsheet CFT with $c = 26$.

There are two ingredients...

(p, q) Minimal CFT (BPZ)

Labelled by $p < q$ relatively prime

$$c = 1 - \frac{6(p-q)^2}{p\,q}$$

Finite set of Virasoro representations

$$\Delta(\mathcal{O}_{r,s}) = \frac{(rq - sp)^2 - (p-q)^2}{4p\,q}$$

$$1 \le r < p\ , \quad 1 \le s < q\ , \quad sp < rq$$

Liouville theory

Worldsheet action:

$$S = \int d^2z \left((\partial\phi)^2 + \mu\, e^{2\,b\,\phi} \right)$$

μ is called the cosmological constant.

Central charge, background charge:

$$c = 1 + 6Q^2, \qquad Q = b + \frac{1}{b}$$

Virasoro primaries:

$$\Delta(e^{2\alpha\phi}) = -\left(\frac{Q}{2} - \alpha\right)^2 + \frac{Q^2}{4}$$

Minimal String Theory

Now combine

(p, q) minimal CFT + Liouville theory

together with the ghosts. Total $c = 26$ sets $b^2 = \frac{p}{q}$

Simplest operators in the BRST cohomology are "tachyons"

$$\mathcal{T}_{r,s} = c\,\bar{c}\,\mathcal{O}_{r,s}e^{2\,\beta_{r,s}\,\phi}$$

$$2\beta_{r,s} = \frac{p+q-(rq-sp)}{\sqrt{p\,q}}$$

$$1 \le r < p\ , \quad 1 \le s < q\ , \quad rq > sp$$

Target space from the worldsheet

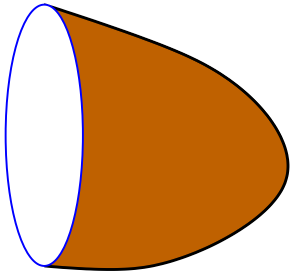
Naively, the classical target space consists of just the Liouville field ϕ.

However, this definition of target space is imprecise, because the worldsheet theory is strongly coupled at $\phi \to +\infty$.

A more precise definition of target space, which can avoid the problems of the strongly-coupled worldsheet, is obtained from the moduli space of D-branes.

For this, we need D-branes with a continuous parameter. Fortunately, Liouville theory supplies us with such D-branes. These are called…

FZZT branes



FZZT branes are labelled by a continuous parameter

$$x = \mu_B$$

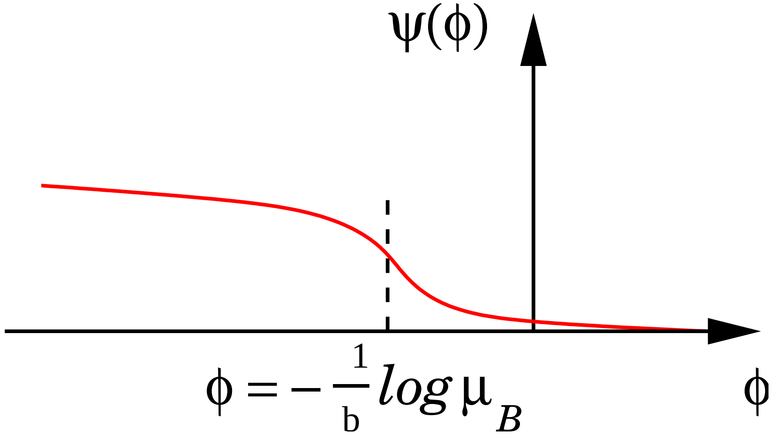
μ_B is called the boundary cosmological constant because it multiplies the boundary interaction

$$\delta S = \mu_B \oint e^{b\phi}$$

which leads to Neumann-like boundary conditions on ϕ.

The minisuperspace wavefunction suggests the brane comes from infinity and dissolves at $\phi \approx -\frac{1}{b}\log\mu_B$.

$$\Psi(\phi) = \langle\phi|\mu_B\rangle = e^{-\mu_B\, e^{b\,\phi}}$$



Thus the tip of FZZT brane acts as a target space probe.

Since the position of the tip is labelled by $x = \mu_B$, we interpret x as a target space coordinate.

Thus the moduli space of FZZT branes $\mathcal{M}$ defines an effective target space.

ZZ branes

Minimal string theory has another kind of D-brane called ZZ branes. There are finitely many, and they are all localized in the $\phi \to +\infty$ region.

They are labelled by integers (m, n) satisfying

$$1 \leq m \leq p - 1, \quad 1 \leq n \leq q - 1, \quad qm - pn > 0$$

ZZ branes correspond to bulk instantons of minimal string theory.

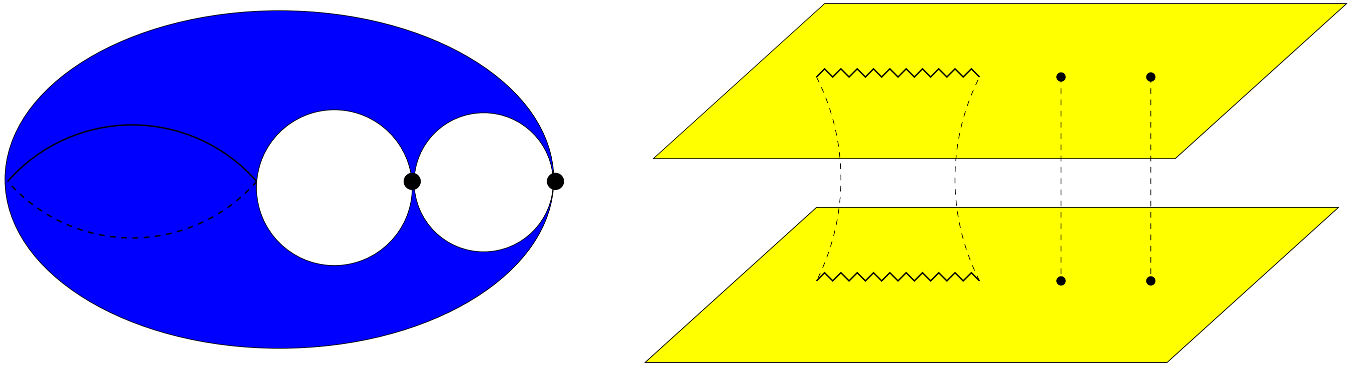
Semiclassical target space

Can view real x as target space. But it is useful to analytically continue to complex plane.

It turns out that perturbative FZZT observables are not single-valued functions of x.

Consider the simplest such observable, the FZZT disk amplitude $Z(x)$. Define

$$y = \partial_x Z(x)$$



Then x and y satisfy an algebraic equation

$$F_{p,q}(x,y) = T_p(y) - T_q(x) = 0$$

This defines a Riemann surface $\mathcal{M}_{p,q}$ which is a p-sheeted cover of the complex x plane.

Properties of $\mathcal{M}_{p,q}$

- $\mathcal{M}_{p,q}$ has $(p-1)(q-1)/2$ singularities (pinched cycles) at the simultaneous solutions of $T_p(y) - T_q(x) = 0$ and $T_p'(y) = T_q'(x) = 0$.

- Apart from the singularities, every point in $\mathcal{M}_{p,q}$ is in one-to-one correspondence with a point $z \in \mathbb{C}$, via

$$(x, y) = (T_p(z), T_q(z))$$

It follows that $\mathcal{M}_{p,q}$ effectively has genus zero.

Generalization to other backgrounds

Other closed-string backgrounds are obtained by turning on physical operators, e.g. the tachyons $\mathcal{T}_{r,s}$.

This deforms $T_p(y) - T_q(x) = 0$ to a more general polynomial equation $F(x, y) = 0$. However, the corresponding Riemann surface still has genus zero.

(Non-zero genus corresponds to adding background ZZ branes.)

Expect $\mathcal{M}_{p,q}$ to persist to all orders in perturbation theory.

What about non-perturbatively?

For this, we need . . .

Dual matrix model description

Minimal string theories are dual to certain large N random matrix models. David, Kazakov, Douglas, Shenker, Gross, Migdal, Brezin…

The modern interpretation is holographic: the matrix model is the theory of open strings between N “condensed” ZZ branes. McGreevy & Verlinde; Klebanov, Maldacena & Seiberg

For instance, the $(p, q) = (2, 2l - 1)$ theories correspond to one-matrix model:

$$Z(g) = \int dM \; e^{-\frac{1}{g}\,\mathrm{Tr}\,V(M)}$$

with M an $N \times N$ Hermitian matrix.

Theories with $p > 2$ can be described using two matrices.

To obtain minimal string theory, we must take the continuum or double-scaling limit. This involves

$$N \to \infty, \quad g \to g_c, \quad N(g - g_c)^{\alpha} = \text{finite}$$

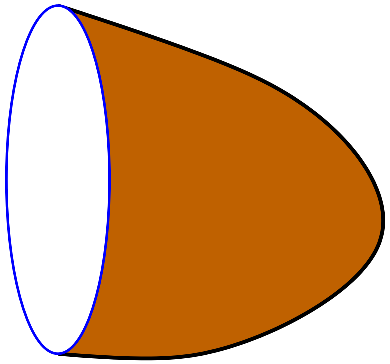
The matrix model can be solved by the method of orthogonal polynomials. Introduce polynomials $P_n(x) = x^n + \dots$ satisfying

$$\int dx\, e^{-\frac{1}{g}V(x)} P_m(x) P_n(x) = h_n \delta_{mn}$$

Then after diagonalizing M, find

$$\begin{aligned} Z(g) &= \int d\lambda \det_{1\le i,j\le N} P_{i-1}(\lambda_j)^2 e^{-\frac{1}{g}\sum_{k=1}^{N} V(\lambda_k)} \\ &= N! \prod_{k=0}^{N-1} h_k \end{aligned}$$

Macroscopic loops vs. D-branes



In the dual matrix model, macroscopic loops (worldsheet boundaries) are created by

$$W(x) = \mathrm{Tr}\log(x - M)$$

E.g. $\langle W(x) \rangle$ corresponds to the FZZT disk amplitude.

Nonperturbatively, D-brane amplitudes must include worldsheets with any number of boundaries (and handles).

In the matrix model, this means we must exponentiate $W(x)$, whereby the D-brane creation operator becomes a determinant

$$e^{W(x)} = e^{\mathrm{Tr}\log(x-M)} = \det(x-M)$$

FZZT partition function

For instance, the exact FZZT partition function is

$$\langle \det(x - M) \rangle = P_N(x)$$

at finite N. Here $P_N(x)$ is the Nth orthogonal polynomial introduced above.

It turns out that $\Psi(x) = e^{-V(x)/2} \det(x - M)$ has a good continuum limit:

$$\langle \Psi(x) \rangle \rightarrow \psi(x)$$

One can show that $\psi(x)$ is an entire function of x.

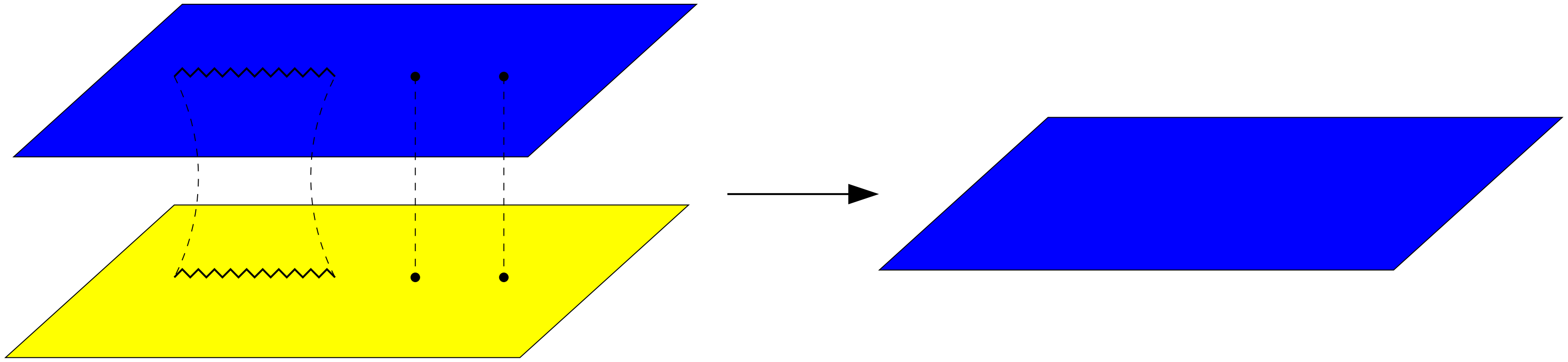
Exact vs. semiclassical target space

Now let us compare the semiclassical and exact answers.

- The semiclassical observables were functions on $\mathcal{M}_{p,q}$, a p-sheeted cover of the complex x plane.

- The exact answer is an entire function on $\mathbb{C}$, a single copy of the x plane.

Therefore, target space is drastically modified by nonperturbative effects!



Example: $(p, q) = (2, 1)$

In this case, the matrix model is the Gaussian matrix model:

$$Z(g) = \int dM \; e^{-\frac{1}{g}\,\mathrm{Tr}\,M^2}$$

The classical target space $\mathcal{M}_{2,1}$ is given by $T_2(y) = T_1(x)$, or

$$y = \sqrt{\frac{x+1}{2}}$$

On the other hand, the finite N FZZT partition function is the Hermite polynomial:

$$\langle \det(x-M) \rangle = \left(\frac{g}{4}\right)^{N/2} H_N(x/\sqrt{g})$$

We can derive this using the following trick. Introduce the Grassmann odd fermions χ_i, $\chi_i^\dagger$, and write

$$\det(x-M) = \int d\chi d\chi^\dagger \, e^{\chi^\dagger (x-M)\chi}$$

Then

$$\begin{aligned}\langle \det(x-M)\rangle &= \int dM d\chi d\chi^\dagger \; e^{-\frac{1}{g}\operatorname{Tr} M^2+\chi^\dagger(x-M)\chi} \\ &= \int d\chi d\chi^\dagger \; e^{-\frac{g}{4}(\chi^\dagger\chi)^2+x\chi^\dagger\chi} \\ &= \int ds d\chi d\chi^\dagger \; e^{-\frac{1}{g}s^2+(is+x)\chi^\dagger\chi} \\ &= \int ds(x+is)^N e^{-\frac{1}{g}s^2}\end{aligned}$$

This is the well-known integral representation of the Hermite polynomials.

Interpretation:

- M describes the open strings between the N condensed ZZ branes.
- χ, $\chi^\dagger$ are the open strings between the FZZT brane and the condensed ZZ branes. They are fermionic.
- s describes an effective degree of freedom on the FZZT brane.

Double scaling limit

The continuum limit is $N = \epsilon^{-3} \to \infty$ with:

$$x \to \sqrt{2}(1 + \epsilon^2 g_s^{-2/3} x),$$

$$g \to \epsilon^{-3}(1 - \epsilon^2 g_s^{-2/3} \tau)$$

Then the FZZT partition function becomes the Airy function:

$$e^{-x^2/2g} \int ds (x + is)^N e^{-\frac{1}{g}s^2} \to \psi(x)$$

$$= \int ds\, e^{is^3/3 + i(x+\tau)s/g_s^{2/3}}$$

$$= Ai((x + \tau) g_s^{-2/3})$$

More on the Airy function

The Airy function is an entire function of x. It has the integral representation:

$$Ai(xg_s^{-2/3}) = \int ds\, e^{(is^3/3+ixs)/g_s}$$

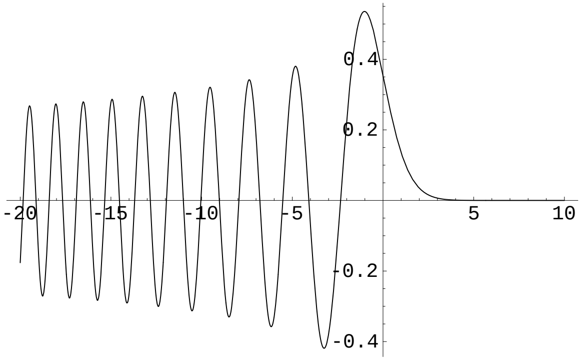
As $g_s \to 0$, we can apply the saddle point approximation.

$$s = \pm\sqrt{-x}$$

are the saddles of the s integral. Thus

$$Ai(xg_s^{-2/3}) \approx A(x)e^{-\frac{2}{3g_s}x^{3/2}} + B(x)e^{\frac{2}{3g_s}x^{3/2}}$$

The two terms correspond to the two sheets of $\mathcal{M}_{2,1}$.



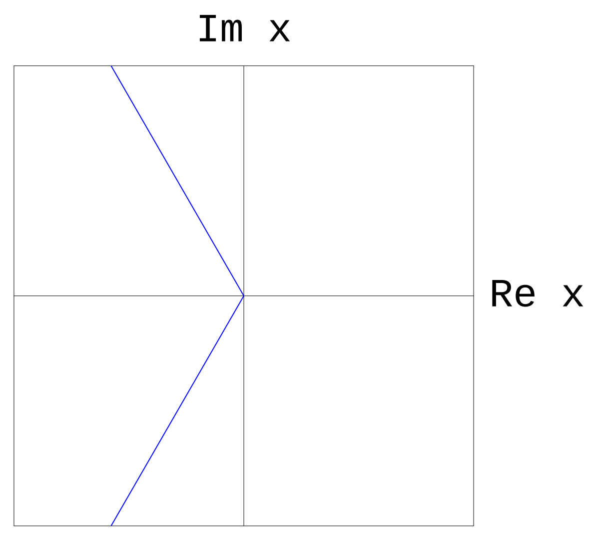
A more detailed analysis shows that:

$$Ai(x) \sim \begin{cases} \frac{1}{2\sqrt{\pi}x^{1/4}} e^{-2/3x^{3/2}} & (x \to +\infty) \\ \frac{1}{\sqrt{\pi}(-x)^{1/4}} \sin\left(\frac{\pi}{4} + \frac{2}{3}(-x)^{3/2}\right) & (x \to -\infty) \end{cases}$$

i.e., the second saddle point is absent for $x > 0$.

Thus the Airy function exhibits Stokes' phenomenon:

$$[\text{analytic continuation, asymptotic expansion}] \neq 0$$



The second saddle abruptly ceases to contribute when we cross the Stokes' lines at $\arg(x) = \pm\frac{2\pi}{3}$.

The Stokes' lines ensure that the second term in $Ai(x) \approx A(x)e^{-\frac{2}{3}x^{3/2}} + B(x)e^{\frac{2}{3}x^{3/2}}$ only contributes when it is nonperturbatively small.

Physical interpretation

We intepret the second term in $A(x)e^{-\frac{2}{3}x^{3/2}} + B(x)e^{\frac{2}{3}x^{3/2}}$ as an instanton effect in the theory on the brane. This follows from thinking of s as the effective degree of freedom on the brane.

This gives a physical interpretation to the second, “unphysical” sheet of $\mathcal{M}_{1,2}$.

It also explains why the second sheet disappears in the exact answer…

…Semiclassically, the different sheets of $\mathcal{M}_{1,2}$ seemed to label different FZZT branes.

However, now we see that they label the same FZZT branes, only with or without instantons. Thus we must sum over the instantons to obtain the exact answer.

General lessons of the exact analysis

- Parts of classical target space $\mathcal{M}_{p,q}$ disappear.
- Some (but not all) regions of $\mathcal{M}_{p,q}$ contribute nonperturbative corrections to the exact answer.
- These corrections can be thought of as instantons in the theory on the brane.
- The general mechanism governing the nonperturbative physics is Stokes' phenomenon.

Relation to other work

Interesting analogies with physics behind BH horizons.

Fidkowski, Hubeny, Kleban & Shenker

In both cases, Stokes' phenomenon and the holographic theory play important roles.

Possible correspondence:

- Outside the horizon ↔ the physical sheet of $\mathcal{M}_{p,q}$
- Behind the horizon ↔ the unphysical sheets of $\mathcal{M}_{p,q}$

Possible lessons for the topological string.

A class topological B-models were recently shown to resemble perturbative minimal string theory.

Aganagic, Dijkgraaf, Klemm, Marino & Vafa

Our analysis suggests non-perturbative modifications to Calabi-Yau?