MPRG the University of Pennsylvania MATH/PHYSICS Research Group

Modular Matrix Models & Monstrous Moonshine:

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Motivations

MATRIX MODELS

- **Resurrection** of old matrix models;
- Dijkgraaf-Vafa Correspondence;
- Powerful unified view of SUSY gauge theory/2D qauntum gravity/geometry;
- Geometrisation and discretisation of string theory;

MOONSHINE

- One of the most amazing “coincidences” in mathematics;
- McKay-Thompson: Relation of elliptic j-function and the Monster Group;
- Conway-Norton: (crazy) Moonshine conjecture;
- Frenkel-Lepowski-Meurman: Vertex Algebras;
- Borcherds: Proof (Fields Medal 98);

QUANTUM/STRINGY MOONSHINE???

- Does moonshine mean anything to String Theory?
- Dixon-Ginsparg-Harvey; Craps-Gaberdiel-Harvey
- Is there a quantum generalisation of moonshine?

Outline

Four Short Pieces

1. The Klein Invariant j-function
2. The One-Matrix Model
3. The Master Field Formalism
4. Dijkgraaf-Vafa

Modular Matrix Models

- Constructing a matrix model given a modular form
- The j-MMM

Discussions and Prospectus

- A precise program for finding quantum corrections
- geometric meaning

Four Short Pieces

I. The Klein Invariant

- Modular Invariant: The most important (only) meromorphic function invariant under $SL(2;\mathbb{Z})$ $(z \to \frac{az+b}{cz+d}\ ,\ ad-bc=1)$ (profound arithmetic properties);

$$q := e^{2\pi i z}\ , \quad \lambda(q) := \left(\frac{\vartheta_2(q)}{\vartheta_3(q)}\right)^4\ , \quad j(e^{2\pi i z}) : \mathcal{H}/SL(2;\mathbb{Z}) \to \mathbb{C}$$

$$j(q) := 1728\ J(q), \qquad J(q) := \frac{4}{27}\,\frac{(1-\lambda(q)+\lambda(q)^2)^3}{\lambda(q)^2(1-\lambda(q))^2}$$

- The q-expansion

$$\begin{aligned} j(q) \quad = \quad & q^{-1} + 744 + 196884\,q + 21493760\,q^2 + 864299970\,q^3 + \\ & 20245856256\,q^4 + 333202640600\,q^5 + 4252023300096\,q^6 \ldots \end{aligned}$$

- j-function and modularity known to Klein, Dedekind, Kronecker, and as far back as Hermite (1859)
- Classification of Simple Groups (1970's)
  Monster = Largest Sporadic Simple Group M,
  $|M| \sim 10^{53} =$
  $2^{46} \cdot 3^{20} \cdot 5^{9} \cdot 7^{6} \cdot 11^{2} \cdot 13^{3} \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 =$
  808017424794512875886459904961710757005754368000000000
- Andrew Ogg (1975): $\mathcal{H}/(\Gamma(p) \subset \Gamma)$,
  $\Gamma(p) := \langle \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2;\mathbb{Z}), c \equiv 0 (\mathrm{mod} p) \right\}, \begin{pmatrix} 1 & 0 \\ -p & 1 \end{pmatrix} \rangle$ has genus $= 0$ if $p = 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 41, 47, 59, 71$
- Jacque Tits (1975): (described the order of the then-conjectural M in a lecture attended by Ogg);

- *Ogg offers a bottle of Jack Daniel's*
- Until JOHN McKAY (1978) Letter to Thompson:
- MOONSHINE (J. Conway and S. Norton)

j-function		Monster
196884	=	$1 + 196883,$
21493760	=	$1 + 196883 + 21296876,$
864299970	=	$2 \cdot 1 + 2 \cdot 196883 + 21296876 + 842609326,$
$\ldots$		

- Frenkel-Lepowski-Meurman: Vertex Algebras (1980's);
- RICHARD BORCHERDS, Proof (1986) Fields (1998)

II. The One-Matrix Model

- The Hermitian one-matrix model (easily generalised to complex)

$$
\begin{aligned}
Z &= \frac{1}{\mathrm{Vol}(U(N))} \int [\mathcal{D}\Phi] \exp\left(-\frac{1}{g}\,\mathrm{Tr}\, V(\Phi)\right), \\
&= \frac{1}{\mathrm{Vol}(U(N))} \int \prod_{i=1}^{N} d\lambda_i\, \Delta(\lambda)^2 \exp\left(-\frac{N}{g} \sum_i V(\lambda_i)\right)
\end{aligned}
$$

Vandermonde: $\Delta(\lambda) = \prod_{i<j} (\lambda_j - \lambda_i)$

- At PLANAR LIMIT (higher genus $\sim 1/N$-expansions):
  - Eigenvalue Density: $\rho(\lambda) := \frac{1}{N} \sum_i \delta(\lambda - \lambda_i)$;
  - Saddle Point: $2 \fint d\tau\, \frac{\rho(\tau)}{\lambda - \tau} = \frac{1}{g} V'(\lambda)$;
- CUTS Branch-cuts $\rightsquigarrow$ solution of integral eq.

- Resolvent: $R(z) = \int d\lambda \, \frac{\rho(\lambda)}{z-\lambda}$ satisies **loop equation**

$$R(z)^2 - \frac{1}{g} V'(z) R(z) - \frac{1}{4g^2} f(z) = 0$$

- Purely algebraic equation.

- Reverse Engineering:

$$\begin{aligned} \rho(z) &= \frac{1}{2\pi i} \lim_{\epsilon \longrightarrow 0} \left(R(z + i\epsilon) - R(z - i\epsilon) \right), \\ -\frac{1}{g} V'(z) &= \lim_{\epsilon \longrightarrow 0} \left(R(z + i\epsilon) + R(z - i\epsilon) \right) \end{aligned}$$

- KEY: knowing $R(z) \rightsquigarrow$ knowing everything about the MM.

III. The Master-Field Formalism

- A convenient (algebraic) formulation of matrix models.
- In the 1MM, observables are $\mathcal{O}_k := \mathrm{Tr}\Phi^k$

$$\langle \mathcal{O}_k \rangle = Z^{-1} \lim_{N\to\infty} \frac{1}{N} \int [\mathcal{D}\Phi]\, \mathrm{Tr}\ \mathcal{O}_k \exp\left(-\frac{N}{g}\mathrm{Tr}\ V(\Phi)\right),$$

- Using **free probability theory**, Voiculescu shewed that correlators of MM are encoded in the CUNTZ algebra

$$aa^\dagger = \mathbb{I}, \quad a^\dagger a = |0\rangle\langle 0|, \qquad \text{with} \quad a|0\rangle = 0$$

and there exists a Master Field $\hat{M}(a, a^\dagger)$ s.t.

$$\langle \mathcal{O}_k \rangle = \langle 0|\hat{M}(a, a^\dagger)^k|0\rangle$$

THM [Voiculescu]: In particular, for the 1MM

$$\hat{M}(a, a^\dagger) = a + \sum_{n=0}^{\infty} m_n (a^\dagger)^n$$

(m_n are coefficients)

- VEV's are in *Voiculsecu polynomials* of m_n:

$$\begin{aligned} \langle \mathcal{O}_1 \rangle &= \mathrm{tr}[M] := \langle \hat{M}(a, a^\dagger) \rangle = m_0, \\ \langle \mathcal{O}_2 \rangle &= \mathrm{tr}[M^2] := \langle \hat{M}(a, a^\dagger)^2 \rangle = m_0^2 + m_1, \\ \langle \mathcal{O}_3 \rangle &= \mathrm{tr}[M^3] := \langle \hat{M}(a, a^\dagger)^3 \rangle = m_0^3 + 3m_0 m_1 + m_2 \end{aligned}$$

- Write generating function

$$K(z) = \frac{1}{z} + \sum_{n=0}^{\infty} m_n z^n$$

then, the resolvent is simply the inverse: $R(z) = K^{-1}(z)$.

- To determine the Voiculescu polynomials, simply series-invert

$$f(z) = \frac{1}{z} + b_0 + b_1 z + b_2 z^2 + b_3 z^3 + b_4 z^4 + \ldots$$

to give

$$\begin{aligned} f^{-1}(z) \quad = \quad & \frac{1}{z} + \frac{b_0}{z^2} + \frac{{b_0}^2 + b_1}{z^3} + \frac{{b_0}^3 + 3\, b_0\, b_1 + b_2}{z^4} + \frac{{b_0}^4 + 6\, {b_0}^2\, b_1 + 2\, {b_1}^2 + 4\, b_0\, b_2 + b_3}{z^5} \\ & + \frac{{b_0}^5 + 10\, {b_0}^3\, b_1 + 10\, b_0\, {b_1}^2 + 10\, {b_0}^2\, b_2 + 5\, b_1\, b_2 + 5\, b_0\, b_3 + b_4}{z^6} + \ldots . \end{aligned}$$

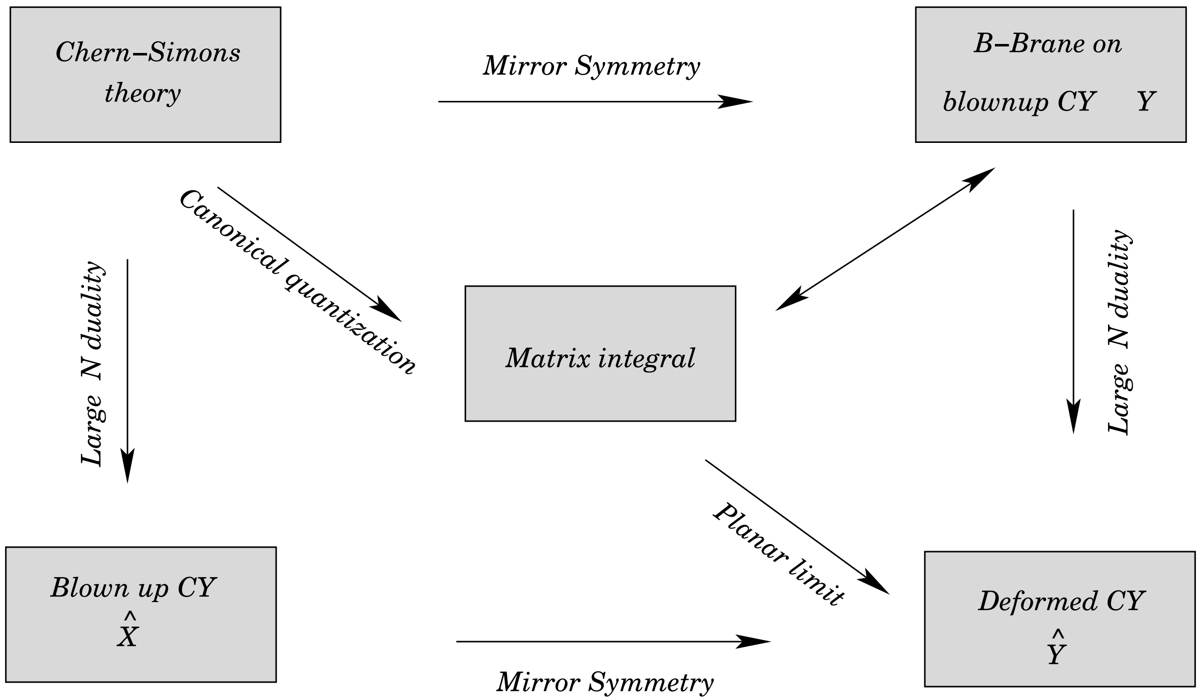
Rmk: (McKay) The Voiculescu polynomials $\sim$ generating function for the number of Dyke paths in a 2-D grid (Catalan Numbers).

- KEY POINT:
  Master Field $\rightsquigarrow$ Resolvent $\rightsquigarrow$ Everything about the MM

Rmk: The formalism become very convenient for multi-matrix models, e.g., QCD

IV. Dijkgraaf-Vafa

- Generalisation and new perspective on the Gopakumar-Vafa **large N duality** for the conifold.
- An intricate web (from Aganagic-Klemm-Mariño-Vafa 0211098)



- an $U(n)$ gauge theory, adjoint Φ and tree-level superpotential

$$W_{\rm tree}(\Phi) = \sum_{k=1}^{p+1} \frac{1}{k} g_k {\rm Tr}\ \Phi^k$$

  - Full non-pert. effective (Cachazo-Intriligator-Vafa) in glueball $\mathcal{S} = \frac{1}{32\pi^2}\ {\rm Tr}\ \mathcal{W}^\alpha \mathcal{W}_\alpha$

$$W_{\rm eff}(\mathcal{S}) = n \frac{\partial}{\partial \mathcal{S}} F_0(\mathcal{S}) + \mathcal{S}(n \log(\mathcal{S}/\Lambda^3) - 2\pi i \tau)$$

  - $F_0(\mathcal{S})$ is the planar free energy of a large N (bosonic) MM with potential $W_{\rm tree}(\Phi)$; identify: $\mathcal{S} \equiv gN$ ('t Hooft)
  - $\mathcal{N} = 1$ thy is geometrically engineered on (local) CY3

$$\{u^2 + v^2 + y^2 + W'_{tree}(x)^2 = f_{p-1}(x)\} \subset \mathbb{C}^4,$$

– Special Geometry: cpt A-cycles and non-cpt B-cycles, identify $\mathcal{S}_i = \int_{A_i} \Omega, \Pi_i := \frac{\partial F_0}{\partial \mathcal{S}_i} = \int_{B_i} \Omega$,
$\left(N_i := \int_{A_i} G_3, \quad \alpha := \int_{B_i} G_3 \right)$

$$\Rightarrow W_{\text{eff}}(\mathcal{S}) = \int_{CY_3} G_3 \wedge \Omega = \sum_{i=1}^{p} N_i \Pi_i + \alpha \sum_{i=1}^{p} \mathcal{S}_i$$

– non-trivial geometry is the hyper-elliptic curve:

$$y^2 = W'_{tree}(x)^2 + f_{p-1}(x)$$

–1. The Seiberg-Witten curve of the $\mathcal{N} = 1$ theory (deformation of $\mathcal{N} = 2$ by W_{tree};
2. The spectral curve (loop eq) of MM

- KEY POINT: Each (bosonic) MM actually computes non-perturbative information for an $\mathcal{N} = 1$ gauge theory geometrically engineered on a CY3.

Modular Matrix Models

Observatio Curiosa:

- q-expansion: $f(q) = q^{-1} + \sum_{n=0}^{\infty} a_n q^n$
- Master Field: $\hat{M}(a, a^\dagger) = a + \sum_{n=0}^{\infty} m_n (a^\dagger)^n$

- Question: *Can we consistently construct a MM whose master field is a given modular form?*
- Take the favourite and most important example:

$$j(q) = \frac{1}{q} + \sum_{n=0}^{\infty} m_n q^n = \frac{1}{q} + m_0 + m_1 q + \ldots$$

$\{m_0, m_1, \ldots, m_5, \ldots\} =$

$\{744, 196884, 21493760, 864299970, 20245856256, 333202640600, \ldots\}$

- Procedure:
  1. Identify $j(q) \sim K(q)$, the generating function for the Master;
  2. Resolvent $R(z) = j^{-1}(e^{2\pi i z})$.
- KEY: Find the inverse of j as a function of z.
- The Inverse j-function (well-known)

$$
\begin{aligned}
j^{-1}(z) &= i\left(\frac{r(z)-s(z)}{r(z)+s(z)}\right), \quad r(z) := \tilde{r}\left(\frac{z}{1728}\right), \quad s(z) := \tilde{s}\left(\frac{z}{1728}\right) \\
\tilde{r}(z) &:= \Gamma\left(\frac{5}{12}\right)^2 \ {}_2F_1\left(\frac{1}{12}, \frac{1}{12}; \frac{1}{2}; 1-z\right), \\
\tilde{s}(z) &:= 2(\sqrt{3}-2)\ \Gamma\left(\frac{11}{12}\right)^2 \sqrt{z-1}\ {}_2F_1\left(\frac{7}{12}, \frac{7}{12}; \frac{3}{2}; 1-z\right).
\end{aligned}
$$

- The Branch-cuts
  - Two-cut: $(-\infty, 0] \cup [1, \infty)$

– For Hypergeometrics:

$$\lim_{\epsilon\to 0} {}_2F_1(a,b;c;z-i\epsilon) = {}_2F_1(a,b;c;z),$$
$$\lim_{\epsilon\to 0} {}_2F_1(a,b;c;z+i\epsilon) =$$
$$\frac{2\pi i e^{\pi i(a+b-c)}\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)\Gamma(a+b-c+1)}\, {}_2F_1(a,b;a+b-c+1;1-z) + e^{2\pi i(a+b-c)}\, {}_2F_1(a,b;c;z)$$

– Discontinuity of the resolvent:

$$R(z+i\epsilon) \pm R(z-i\epsilon) = \begin{cases} i\,\frac{e^{\frac{\pi i}{3}}(s-r)+(t-u)}{-e^{\frac{\pi i}{3}}(s+r)+(t+u)} \pm i\,\frac{r-s}{r+s}, & z\in(-\infty,0); \\ (1\pm 1)\,i\,\frac{r-s}{r+s}, & z\in(0,1); \\ i\,\frac{r-s}{r+s} \pm i\,\frac{r+s}{r-s}, & z\in(1,\infty). \end{cases}$$

- KEY: have analytic form for the resolvent
- Recall:

$$\rho(z) = \frac{1}{2\pi i}\lim_{\epsilon\to 0}\left(R(z+i\epsilon) - R(z-i\epsilon)\right)$$
$$-\frac{1}{g}V'(z) = \lim_{\epsilon\to 0}\left(R(z+i\epsilon) + R(z-i\epsilon)\right)$$

Constructing the MMM

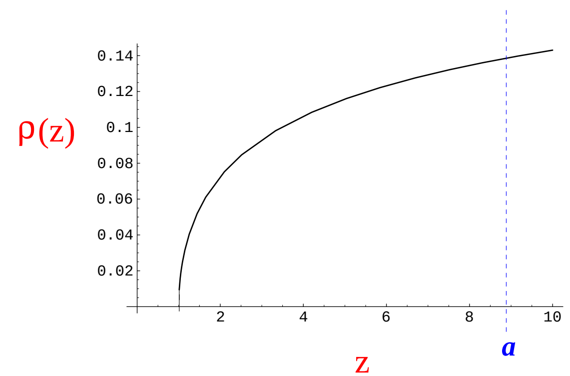
- The eigenvalue distribution:

$$\rho(z) = \begin{cases} \frac{1}{\pi}\left(\frac{st-ru}{(r+s)(t+u-e^{\frac{\pi i}{3}}(r+s))}\right), & z \in (-\infty, 0); \\ 0, & z \in (0,1); \\ \frac{1}{\pi}\left(\frac{2rs}{s^2-r^2}\right), & z \in (1,\infty). \end{cases}$$

- real for $z \in [1,\infty)$, so for convenience restrict to this range (similar restriction done in **Gross-Witten** model) where MM is Hermitian (Rmk: [Lazaroiu] need $\mathbb{C}$-MM for DV)
- Normalisation and regularisation: $\lim_{a\to\infty} A(a) \int_1^a dz\, \rho(z) = 1.$

- Will thus take

$$\rho(z) \sim \begin{cases} 0, & z < 1; \\ \frac{1}{\pi}\left(\frac{2rs}{s^2 - r^2}\right), & z \in (1, a). \end{cases}$$



- our MMM: 1-cut Hermitian 1MM
- Similarly, the potential of the MMM is $(z \in (1, a))$

$$-\frac{1}{g}V'(z) \sim \left(i\frac{r-s}{r+s} + i\frac{r+s}{r-s}\right) = j^{-1}(e^{2\pi i z}) + \frac{1}{j^{-1}(e^{2\pi i z})}$$

- Planar Free-Energy:

$$
\begin{aligned}
F_0 &= \int dx\, \rho(x) V(x) - \int\int dx\, dy\, \rho(x)\rho(y) \log|x-y| \\
&= \int dx\, \rho(x) \left(\tfrac{1}{2} V(x) - \log x\right)
\end{aligned}
$$

- Alternatives, e.g., $1/(q^2 j(q))$; *made the simplest choice that is analytically invertible.*

- KEY POINT: **CAN** consistently construct a (1-cut Hermitian) MM whose master field corresponds to the j-function.

Salient Features

Change back to q-expansion (from Laurent in z), integers emerge:

- What are the observables?

$$\begin{aligned}\langle \mathrm{Tr}\Phi\rangle = 744 &= 744 = m_0;\\ \langle \mathrm{Tr}\Phi^2\rangle = 750420 &= 196884 + 744^2 = m_1 + m_0^2;\\ \langle \mathrm{Tr}\Phi^3\rangle = 872769632 &= 21493760 + 3\cdot 744\cdot 196884 + 744^3 = m_2 + 3m_0m_1 + m_0^3\end{aligned}$$

- VEV's = Voiculescu polynomials in coef($j(q)$) $\Rightarrow$ polynomials p_n in irrep(Monster)
- **Question:** Are p_n interesting polynomials?
- RMK: Can one construct MM whose VEV's are directly coef($j(q)$)?

1. We want:

$$\begin{aligned} \mathrm{tr}[\Phi] &= \mu_0 = 744; \\ \mathrm{tr}[\Phi^2] &= \mu_1 + \mu_0^2 = 196884; \\ \mathrm{tr}[\Phi^3] &= \mu_2 + 3\mu_0\mu_1 + \mu_0^3 = 21493760; \ldots \end{aligned}$$

2. i.e., $j(q) = \frac{1}{q} + \mu_0 + (\mu_1 + \mu_0^2)q + (\mu_2 + 3\mu_0\mu_1 + \mu_0^3)q^2 + \ldots$
3. Can invert analytically: $\frac{1}{q^2}\, j(1/q) = \frac{1}{q} + \frac{\mu_0}{q^2} + \frac{\mu_1+\mu_0^2}{q^3} + \frac{\mu_2+3\mu_0\mu_1+\mu_0^3}{q^4} + \ldots$
4. Resolvent is $\frac{1}{q^2}\, j(1/q) = M(q)^{-1}$
5. Gives a $\mathbb{C}$-MM

- q-expansion of the potential:

$$V(q) \sim q - 744 \log q + \frac{196883}{q} + \frac{83987356}{q^2} + \frac{60197568710}{q^3} + \ldots$$

- **Question:** Are these integers of any interest?

- Dijkgraaf-Vafa Perspective:

  – Potential is simply W_{tree}:

$$W_{\text{tree}} = V(z) \sim \int^z j^{-1}(e^{2\pi i z}) + \frac{1}{j^{-1}(e^{2\pi i z})}$$

  – Can thus get full non-pert.

$$\begin{aligned} W_{\text{eff}}(\mathcal{S}) &= n\frac{\partial}{\partial \mathcal{S}} F_0(\mathcal{S}) + \mathcal{S}(n\log(\mathcal{S}/\Lambda^3) - 2\pi i\tau) \ , \\ F_0(\mathcal{S} = gN) &= \int^z \rho(z)\left(\frac{1}{2}V(z) - \log z\right) \end{aligned}$$

  – Unfortunately, only *numerical*

  – Question: Is W_{eff} a modular form??? If so, it is a quantum version of $j(q)$???

  – need exact analytic expression for planar free energy...

– The CY3 on which the theory is geometrically engineered is

$$\{u^2 + v^2 + s(x, y) = 0\} \subset \mathbb{C}[x, y, u, v]$$

with spectral curve

$$s(x, y) = y^2 - V'(x)^2 - \frac{1}{4g^2} f(x) = 0$$

here $V'(z) \sim j^{-1}(e^{2\pi i z}) + \frac{1}{j^{-1}(e^{2\pi i z})}$ and

$$f(z) \sim 1 + (j^{-1}(e^{2\pi i z}))^2 + i \int_1^\infty \frac{d\lambda}{z - \lambda} \left[\frac{1}{\pi} \frac{4rs(r^2 + s^2)}{(s^2 - r^2)^2} \right]$$

– RMK: *Don't be bothered* by Laurent superpotential and the geometry (non-algebraic), similar analysis was done for Gross-Witten cosine model.

Discussions and Prospectus

Two previous avatars?

1. "Beauty and the Beast" (Dixon-Ginsparg-Harvey):
   The modular invariant torus partition function for bosonic closed string theory on $T^{24} = \mathbb{R}^{24}/\Lambda_{\text{Leech}}$
   (Λ_{Leech} = Leech lattice, unique even, self-dual lattice in 24d with no points of length-squared 2)

$$Z_{\text{Leech}}(q) = \frac{\Theta_{\Lambda_{\text{Leech}}}(q)}{\eta(q)^{24}} = \frac{\sum_{\beta \in \Lambda_{\text{Leech}}} q^{\frac{1}{2}\beta^2}}{\eta(q)^{24}} = j(q) + \text{const.}$$

   Monster Group = $\text{Aut}(\text{CFT}/\mathbb{Z}_2)$

2. An old observation that the K3 surface given by the elliptic fibration

$$y^2 = 4x^3 - \frac{27s}{s-1}x - \frac{27s}{s-1}, \quad s \in \mathbb{P}^1$$

has the mirror map

$$z(q) = \frac{1}{j(q)}$$

(cf. Tian-Yau, Doran: Modularity of Mirror Maps for K3 and relation to Hauptmoduls of the Monster.)

We have

- Been inspired by the formal resemblance between q-expansions of modular forms and Cuntz-expansions of Master Fields
- Consistently constructed a MMM whose master is the elliptic $j(q)$, the potential is

$$V(z) \sim j^{-1}(e^{2\pi i z}) + \frac{1}{j^{-1}(e^{2\pi i z})}$$

- Observables are *per construtio* Voiculescu polynomials in coef($j(q)$), and polynomials p_n in dimIrrep(Monster)
- Q: Are p_n significant?
- Eluded to a CMMM whose observables directly encode dimIrrep(Monster).
- Using Dijkgraaf-Vafa, we have a CY3 with special geometry and a spectral hyper-elliptic curve intimately tied to the MMM.
  - it geometrically engineers an $\mathcal{N} = 1$ theory with $W_{\text{tree}} = V(z)$
  - can find full effective action of the theory
  - Q: Does W_{eff} have interesting modular properties? A quantum generalisation of $j(q)$.

– The CY3 is given by

$$\{u^2 + v^2 + s(x,y) = 0\} \subset \mathbb{C}[x,y,u,v] \quad \text{with}$$

$$s(x,y) = y^2 - \left(j^{-1}(e^{2\pi i z}) + \frac{1}{j^{-1}(e^{2\pi i z})}\right)^2 - \frac{1}{4g^2} f(x) = 0$$

– Q: How is this hyper-elliptic curve related to the Tian-Yau K3? Which K3 gives $j(q)$ rather than its reciprocal?

– Q (Iqbal): What are GW-invariants associated with this CY3? arithmetic properties?

– DV Correspondence as a perturbative window to number theory?

- QUANTUM MOONSHINE?
  - Everything is done to planar limit ($g = 0$), all the saddle point equations, free-energy etc.
  - Next order, i.e., $\mathcal{O}(1/N)$, is well-known (ambjorn, Makeenko et al.), will get "natural" corrections to coef($j(q)$) order by order.
  - Are these integers? What do they count?
- How does the Monster CFT play a rôle?
- Other modular forms? Technique of MMM is general.

R. Borcherds On Moonshine...

I sometimes wonder if this [proving Moonshine] is the feeling you get when you take certain drugs. I don't actually know, as I have not tested this theory of mine."