Instantons in gauge theories with N=1/2 supersymmetry

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Outline

- Noncommutative superspace.
- Gauge theory on NS
  - classical aspects
  - perturbative regime
  - instanton solutions and supersymmetry
- One instanton solution in U(N) gauge theory.
  - procedure for deforming the instanton
  - geometry of the deformed moduli space
- Chiral ring and gluino condensate
- Summary

String in graviphoton field

- String in flat space

$$L = \frac{1}{\alpha'}\left(\frac{1}{2}\tilde{\partial}x^\mu \partial x_\mu + p_\alpha \tilde{\partial}\theta^\alpha + \bar{p}_{\dot{\alpha}}\tilde{\partial}\bar{\theta}^{\dot{\alpha}} + \tilde{p}_\alpha \partial\tilde{\theta}^\alpha + \tilde{\bar{p}}_{\dot{\alpha}}\partial\tilde{\bar{\theta}}^{\dot{\alpha}}\right)$$

Berkovits '96

- Euclidean target space: independent $\theta, \bar{\theta}, p, \bar{p}$:

$$(p_\alpha, \bar{p}_{\dot{\alpha}}) \to \left(-\frac{\partial}{\partial\theta^\alpha}, -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}\right)\Big|_x, \quad q_\alpha \to \frac{\partial}{\partial\theta^\alpha}\Big|_y, \quad \bar{d}_{\dot{\alpha}} \to -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}\Big|_y$$

$$y^\mu = x^\mu + i\theta^\alpha \sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}} + i\tilde{\theta}^\alpha \sigma^\mu_{\alpha\dot{\alpha}}\tilde{\bar{\theta}}^{\dot{\alpha}}$$

- Change of variables: $p \longrightarrow q,\ \bar{p} \longrightarrow \bar{d}$:

$$q_\alpha = -p_\alpha - i\sigma^\mu_{\alpha\dot{\alpha}}\partial x_\mu + \frac{1}{2}\bar{\theta}\theta\partial\theta_\alpha - \frac{3}{2}\partial(\theta_\alpha\bar{\theta}\theta)$$

- New Lagrangian:

$$L = \frac{1}{\alpha'}\left(\frac{1}{2}\tilde{\partial}y^\mu \partial y_\mu - q_\alpha \tilde{\partial}\theta^\alpha + \bar{d}_{\dot{\alpha}}\tilde{\partial}\bar{\theta}^{\dot{\alpha}} - \tilde{q}_\alpha \partial\tilde{\theta}^\alpha + \tilde{\bar{d}}_{\dot{\alpha}}\partial\tilde{\bar{\theta}}^{\dot{\alpha}}\right)$$

- D brane: $\theta = \tilde{\theta},\ q = \tilde{q}$ at $z = \tilde{z}$
- Preserved SUSY:

$$\oint (q dz + \tilde{q} d\tilde{z}), \qquad \oint (\bar{q} dz + \tilde{\bar{q}} d\tilde{z})$$

Graviphoton and Noncommutative Superspace

- Adding graviphoton field to the Lagrangian:

$$L_1 = \frac{1}{\alpha'} \left(-q_\alpha \tilde{\partial} \theta^\alpha - \tilde{q}_\alpha \partial \tilde{\theta}^\alpha + \alpha' F^{\alpha\beta} q_\alpha \tilde{q}_\beta \right)$$

- To avoid gravitational backreaction: $F_{\dot{\alpha}\dot{\beta}} = 0$

- Effective Lagrangian:

$$L_{eff} = \left(\frac{1}{\alpha' F} \right)_{\alpha\beta} \partial \tilde{\theta}^\alpha \tilde{\partial} \theta^\beta$$

- Boundary conditions at $z = \tilde{z}$:

$$\left(\frac{1}{\alpha' F} \right)_{\alpha\beta} (\partial \tilde{\theta}^\alpha \delta \theta^\beta + \tilde{\partial} \theta^\alpha \delta \tilde{\theta}^\beta) = 0 : \qquad \begin{array}{c} \theta^\alpha = \tilde{\theta}^\alpha, \\ \partial \tilde{\theta}^\alpha = -\tilde{\partial} \theta^\alpha \end{array}$$

- Propagators:

$$\langle \theta^\alpha(z, \tilde{z}) \, \theta^\beta(w, \tilde{w}) \rangle = \frac{\alpha'}{2\pi i} F^{\alpha\beta} \log \frac{\tilde{z} - w}{z - \tilde{w}}$$

$$\langle \theta^\alpha(\tau) \, \theta^\beta(\tau') \rangle = \frac{\alpha'}{2} F^{\alpha\beta} \text{sign}(\tau - \tau')$$

$$\{\theta^\alpha, \theta^\beta\} = {\alpha'}^2 F^{\alpha\beta} = C^{\alpha\beta}, \qquad [y^\mu, y^\nu] = 0$$

Seiberg '03

Gauge theory on Noncommutative Superspace

- Noncommutative superspace:

$$\{\theta^\alpha, \theta^\beta\} = C^{\alpha\beta}, \qquad y^m = x^m + i\theta^\alpha \sigma^m_{\alpha\dot\alpha} \bar\theta^{\dot\alpha}$$

- Star product: finite number of terms

$$f(\theta) \star g(\theta) = f(\theta) \exp\left(-\frac{C^{\alpha\beta}}{2} \frac{\overleftarrow{\partial}}{\partial\theta^\alpha} \frac{\overrightarrow{\partial}}{\partial\theta^\beta} \right) g(\theta)$$

- Modification of SUSY algebra:

$$\{\bar Q_{\dot\alpha}, \bar Q_{\dot\beta}\}_\star = -4C^{\alpha\beta} \sigma^m_{\alpha\dot\alpha} \sigma^n_{\beta\dot\beta} \frac{\partial^2}{\partial y^m \partial y^n}$$

- Gauge field:

$$W_\alpha = -\frac{1}{4} \overline{DD} \left(e_\star^{-V} \star D_\alpha e_\star^V \right) \to e_\star^{-i\Lambda} \star W_\alpha \star e_\star^{i\Lambda}$$

- WZ gauge: C–dependent corrections to V.

- Action for "$\mathcal{N} = \frac{1}{2}$" SYM:

$$S = -\int d^4x \mathrm{Tr} \left[\frac{i\tau}{8\pi} W^\alpha \star W_\alpha \right]_{\theta^2} + \int d^4x \mathrm{Tr} \left[\frac{i\overline{\tau}}{8\pi} \overline{W}_{\dot\alpha} \star \overline{W}_{\dot\alpha} \right]_{\bar\theta^2}$$

Perturbative $\mathcal{N} = \frac{1}{2}$ SYM

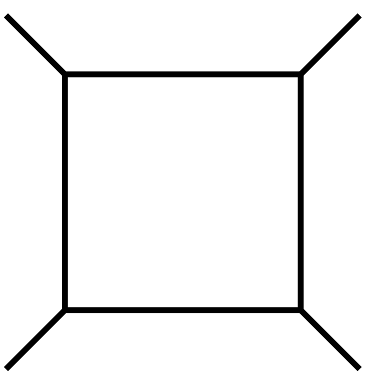
- Lagrangian for the component fields

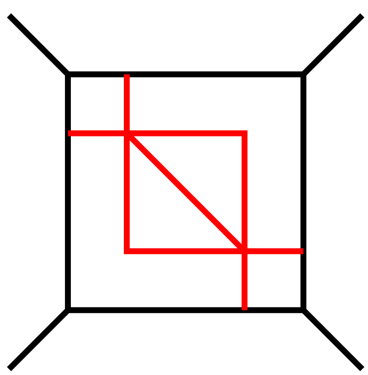
$$L = \frac{1}{g^2}\text{Tr}\left\{-\frac{1}{4}F_{mn}F^{mn} - i\bar{\lambda}\bar{\sigma}^m\nabla_m\lambda + \frac{1}{2}D^2\right\}$$
$$+\frac{1}{g^2}\text{Tr}\left\{-\frac{iC^{mn}}{2}F_{mn}\overline{\lambda\lambda} + \frac{C^2}{8}(\overline{\lambda\lambda})^2\right\}$$

Seiberg '03

- Operators with $\Delta = 5$: no renormalizability

$$\Omega_{\text{div}} = 4 + \sum_{i\in L}(\Delta_i - 4) - \frac{1}{2}\sum_{\text{ext}}(r_l + d_l + 4)$$
$$= 4 + \sum_{i\in L}(\Delta_i - 4) - \sum_{\text{ext}} s_l$$

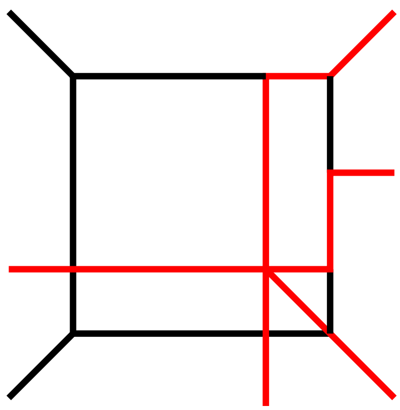




- Assumption: new vertices are connected without changing external lines

Renormalization of $\mathcal{N} = \frac{1}{2}$ SYM

- Features of the theory:
  - no hermiticity
  - R symmetry: $\lambda \to e^{i\alpha}\lambda$
- "Non–renormalizable" vertices: lines cannot terminate inside the diagram



$$\Omega_{\text{div}} = 4 + \sum_{i \in L}(\Delta_i - 4) + {\sum_{i \in L}}'(\Delta_i - 4) - \sum_{\text{ext}} s_l - {\sum_{\text{ext}}}' s_l$$
$$= 4 + \sum_{i \in L}(\Delta_i - 4) + {\sum_{i \in L}}'(\tilde{\Delta}_i - 4) - \sum_{\text{ext}} s_l$$

- $\tilde{\Delta} < 4$ accounts for R charge flow
- SYM is renormalizable: no new vertices.

O L, Rey

Instantons in $\mathcal{N}=\frac{1}{2}$ SYM

- Instantons in $\mathcal{N}=1$ SYM
  - minimal action in a given topological sector
  - solutions preserving $\mathcal{N}=\frac{1}{2}$ SUSY
- SUSY transformations in $\mathcal{N}=\frac{1}{2}$ theory

$$\delta\lambda^{\alpha} = i\varepsilon^{\alpha} D + 2\Big(F^{\alpha\beta} + \frac{i}{2}C^{\alpha\beta}\overline{\lambda\lambda}\Big)\varepsilon_{\beta}$$

$$\delta F_{\alpha\beta} = -i\varepsilon_{(\alpha}\nabla_{\beta)\dot{\beta}}\overline{\lambda}^{\dot{\beta}} \qquad \delta D = -\varepsilon^{\alpha}\nabla_{\alpha\dot{\beta}}\overline{\lambda}^{\dot{\beta}}$$

- Instantons preserving SUSY

$$F^{\alpha\beta} + \frac{i}{2}C^{\alpha\beta}\overline{\lambda\lambda} = 0, \qquad \nabla_{\alpha\dot{\beta}}\overline{\lambda}^{\dot{\beta}} = 0, \qquad \lambda_{\alpha} = D = 0$$

- Alternative derivation: rewrite the action as

$$S = \frac{1}{g^2}\int d^4x \mathrm{Tr}\left[-\left(F^{(+)}_{mn} + \frac{i}{2}C_{mn}\,\overline{\lambda\lambda}\right)^2 - i\lambda\,\sigma^m\nabla_m\bar{\lambda} + D^2\right]$$
$$-\frac{i\overline{\tau}}{4\pi}\int \mathrm{Tr}F\wedge F.$$

- "Instanton number" is negative
- "Holomorphic instanton" – no deformation:

$$F^{\dot{\alpha}\dot{\beta}} = 0, \qquad \nabla_{\dot{\alpha}\beta}\lambda^{\beta} = 0, \qquad \overline{\lambda}_{\dot{\alpha}} = D = 0$$

Constructing Deformed Instantons

- Equations to be solved

$$F^{\alpha\beta} + \frac{i}{2} C^{\alpha\beta} \overline{\lambda\lambda} = 0, \qquad \nabla_{\alpha\dot{\beta}} \overline{\lambda}^{\dot{\beta}} = 0$$

- Perturbation theory in $C_{\alpha\beta}$: truncated series
- Example: one instanton for $U(2)$

$$\bar{\lambda}_{\dot{\alpha}} = F_{\dot{\alpha}\dot{\beta}} \bar{\xi}^{\dot{\beta}}, \qquad \bar{\xi}^{\dot{\alpha}} = \bar{\zeta}^{\dot{\alpha}} + x^{\dot{\alpha}}_{\alpha} \eta^{\alpha}$$

  - Fermi statistics: $\overline{\lambda\lambda} \in U(1)$
  - prepotential for the $U(1)$ part: $A_m = C_{mn} \nabla_n \Phi$
  - solution of the Laplace equation for Φ:

$$\Phi = -8i \left[\frac{\rho^2}{(r^2+\rho^2)^2} \overline{\xi}_{\dot{\alpha}} \overline{\xi}^{\dot{\alpha}} + \frac{1}{r^2+\rho^2} (\overline{\zeta}_{\dot{\alpha}} \overline{\zeta}^{\dot{\alpha}} + \rho^2 \eta^{\alpha} \eta_{\alpha}) \right]$$

Imaanpur

One Instanton for U(N)

- k instantons for $U(N)$: $2kN$ zero modes
- Generically series terminates at $|C|^{kN}$
- One instanton solution: series up to $|C|^3$
- Zero modes for one instanton:

$$\bar{\lambda}^{(0)}_{\dot{\alpha}} = F^{(0)}_{\dot{\alpha}\dot{\beta}}\left(\bar{\zeta}^{\dot{\beta}} + x^{\dot{\beta}}_{\alpha}\eta^{\alpha}\right)$$

$$\bar{\lambda}^{(0)}_{\dot{\alpha}a}{}^{i} = \frac{\varepsilon_{a\dot{\alpha}}\chi^{i}}{(x^2+\rho^2)^{3/2}} \qquad \bar{\lambda}^{(0)}_{\dot{\alpha}i}{}^{a} = \frac{\bar{\chi}^{i}}{(x^2+\rho^2)^{3/2}}\delta^{a}_{\dot{\alpha}}$$

- Global $U(N-2)$ rotation: $\chi_4 = \ldots = \chi_N = 0$
- Exact solution found by perturbation theory

$$A_m = A^{(0)}_m + C_{mn}\nabla_n\left(\Phi^{(1)}+\Phi^{(2)}+\Phi^{(3)}\right) + \frac{i}{16}C_{kl}C_{kl}\left[\Phi^{(1)}, \nabla_n\Phi^{(1)}\right]$$

$$\overline{\lambda}^{\dot{\alpha}} = \overline{\lambda}^{(0)\dot{\alpha}} + \bar{\sigma}^{m\dot{\alpha}\alpha}C_{\alpha}{}^{\beta}\nabla_m\left(\Psi^{(1)}_{\beta} + \Psi^{(2)}_{\beta}\right) - \frac{C_{kl}C^{kl}}{32}\left[\Phi^{(1)}, \left[\Phi^{(1)}, \overline{\lambda}^{(0)\dot{\alpha}}\right]\right]$$

- Poisson equations for prepotentials

$$\nabla^2\Phi^{(m)} = J^{(m)}, \qquad \nabla^2\Psi^{(m)}_{\alpha} = K^{(m)}_{\alpha}$$

Explicit Form of the Instanton

- Undeformed solution

$$(A^{(0)}_{\beta\dot\beta})_c{}^b = -\frac{2i\varepsilon_{ca}}{x^2+\rho^2}(\delta^a_{\dot\beta}x^b_\beta + \delta^b_{\dot\beta}x^a_\beta)$$

$$(F^{(0)}_{\dot\alpha\dot\beta})_c{}^b = \frac{8i\varepsilon_{ca}\rho^2}{(x^2+\rho^2)^2}(\delta^a_{\dot\alpha}\delta^b_{\dot\beta} + \delta^b_{\dot\alpha}\delta^a_{\dot\beta})$$

- Prepotentials

$$(\Phi^{(1)})_a{}^b = -8i\left[\frac{\rho^2}{(r^2+\rho^2)^2}\overline{\xi}_{\dot\alpha}\overline{\xi}^{\dot\alpha} + \frac{1}{r^2+\rho^2}(\overline{\zeta}_{\dot\alpha}\overline{\zeta}^{\dot\alpha} + \rho^2\eta^\alpha\eta_\alpha) - \frac{1}{r^2+\rho^2}\frac{\overline{\chi}_i\chi^i}{64\rho^2}\right]\delta^b_a$$

$$(\Phi^{(1)})_a{}^i = -\frac{2\overline{\xi}_{\dot\alpha}\chi^i}{(r^2+\rho^2)^{3/2}};\quad (\Phi^{(1)})_i{}^a = -\frac{2\overline{\chi}_i\overline{\xi}^{\dot\alpha}}{(r^2+\rho^2)^{3/2}};\quad (\Phi^{(1)})_i{}^j = \frac{i}{r^2+\rho^2}\frac{\bar\chi_i\chi^j}{4\rho^2}$$

$$\begin{aligned}(\Phi^{(2)})_a{}^b = {} & -2iC_{mk}\frac{1}{(\rho^2+r^2)^3}(\overline{\sigma}^{kn})^b{}_a\frac{x_m x_n\overline{\chi}_i\chi^i}{\rho^2} \\ & \times\left[\frac{\overline{\zeta}_{\dot\alpha}\overline{\zeta}^{\dot\alpha}}{\rho^2}(r^2+2\rho^2) - \rho^2\eta^\alpha\eta_\alpha - \eta^\alpha x_{\alpha\dot\alpha}\overline{\zeta}^{\dot\alpha}\right] \\ & + 2i\frac{1}{(\rho^2+r^2)^2}\frac{\overline{\chi}_i\chi^i}{\rho^2}\left[\overline{\zeta}_a(\overline{x}C)^{b\alpha}\eta_\alpha + \overline{\zeta}^b(\overline{x}C)_a{}^\alpha\eta_\alpha\right]\end{aligned}$$

$$(\Phi^{(2)})_i{}^a = -8\frac{\overline{\chi}_i}{(r^2+\rho^2)^{5/2}}\left[(r^2+2\rho^2)\frac{\overline{\zeta}_{\dot\alpha}\overline{\zeta}^{\dot\alpha}}{\rho^2}(\overline{x}C)^{a\alpha}\eta_\alpha + \eta^\alpha\eta_\alpha(\overline{x}Cx)^a{}_{\dot\alpha}\overline{\zeta}^{\dot\alpha}\right]$$

$$(\Phi^{(2)})_a{}^i = -8\frac{\chi^i}{(r^2+\rho^2)^{5/2}}\left[(r^2+2\rho^2)\frac{\overline{\zeta}_{\dot\alpha}\overline{\zeta}^{\dot\alpha}}{\rho^2}(\overline{x}C)_a{}^\alpha\eta_\alpha + \eta^\alpha\eta_\alpha(\overline{x}Cx)_{a\dot\alpha}\overline{\zeta}^{\dot\alpha}\right]$$

$$\Phi^{(3)} = i\frac{C_{kl}C_{kl}}{2}\frac{\eta^\alpha\eta_\alpha\bar\zeta_{\dot\alpha}\bar\zeta^{\dot\alpha}\bar\chi_i\chi^i}{\rho^4(r^2+\rho^2)^3}(r^4+6r^2\rho^2+3\rho^4)\,\mathrm{diag}\left(1,1,2\frac{r^4+4r^2\rho^2+\rho^4}{r^4+6r^2\rho^2+3\rho^4}\right)$$

Metric on the Moduli Space

- Motivation
  - measure on the moduli space
  - metric on MS and AdS/CFT
  - instanton MS in large N $\longrightarrow$ bulk geometry
  - leading contribution: SU(2) instantons

Dorey at '96

- Problems with L^2 metric
  - no manifest gauge invariance
  - no conformal invariance
- Information metric:

$$\mathcal{G}_{AB}\, dZ^A dZ^B \equiv dZ^A dZ^B \int d^4x \, \frac{\partial_A \mathcal{F} \partial_B \mathcal{F}}{\mathcal{F}}$$

Hitchin '88

- Instanton density

$$\mathcal{F}[x, Z^A] = \frac{1}{16\pi^2} \text{Tr}\, F \wedge F$$

- AdS/CFT: bulk–to–boundary propagator:

$$\Delta_Z \mathcal{F}[x, Z^A] = 0, \qquad \mathcal{F}[x, Z^A]\big|_{\rho=0} = \delta^4(x - X)$$

Balasubramanian et al '98

Information Metric

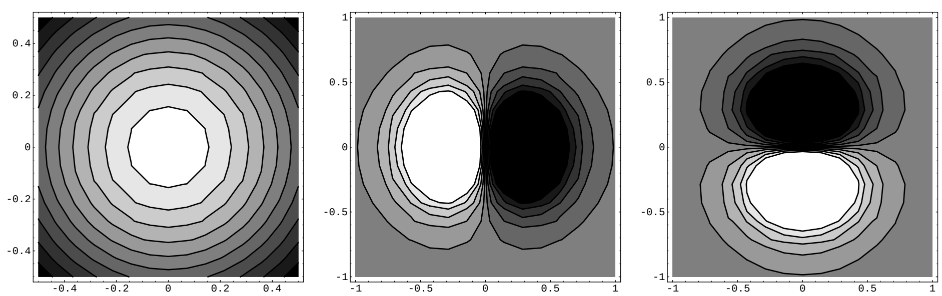
- Undeformed $U(2)$ instanton
  - moduli space: $\rho, X, \bar{\zeta}, \eta$
  - instanton density and information metric

$$\mathcal{F} = \frac{1}{16\pi^2} \frac{96\rho^4}{[(x-X)^2+\rho^2]^4}$$

$$\mathcal{G}_{AB} dZ^A dZ^B = \frac{128}{5}\left[\frac{d\rho^2}{\rho^2} + \frac{dX^2}{\rho^2}\right]$$

Blau, Narain, Thompson '01

- Instanton with C deformation
  - density is a function of $\rho, X, \bar{\zeta}, \eta, \chi, \bar{\chi}$



  - information metric

$$\mathcal{G}_{AB} dZ^A dZ^B = \frac{128}{5}\left[\frac{d\widetilde{\rho}^2}{\widetilde{\rho}^2}\left(1 + \frac{6}{7\widetilde{\rho}^6}C^2 S_1 - \frac{96}{7\widetilde{\rho}^2}C^2 S_2\right)\right.$$

$$\left. + \frac{d\widetilde{X}^2}{\widetilde{\rho}^2}\left(1 - \frac{3}{14\widetilde{\rho}^6}C^2 S_1 + \frac{24}{7\widetilde{\rho}^2}C^2 S_2\right) - C^2 \frac{13}{14\widetilde{\rho}^7} T^m d\widetilde{\rho}\, dX_m\right]$$

  - determinant is C–independent

Chiral Ring & Gluino Condensate

- Antichiral ring: $[Q, \overline{\mathcal{O}}] = 0$.
  - ring property $\overline{\mathcal{O}} \sim \overline{\mathcal{O}} + [Q, M\}$:

$$\langle [Q, M\} \overline{\mathcal{O}}_1 \ldots \overline{\mathcal{O}}_n \rangle = 0$$

  - coordinate independence:

$$\partial \overline{\mathcal{O}} \sim [Q, [\overline{Q}, \overline{\mathcal{O}}\}\}$$

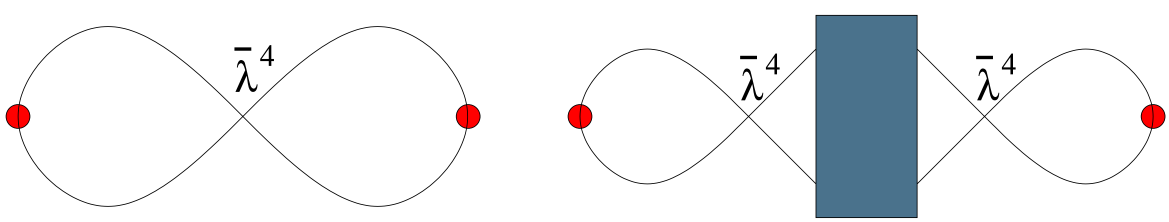
  - C–independence

$$\delta L = \{Q^\alpha, J_\alpha\} \qquad \frac{\delta}{\delta C^{\alpha\beta}} \langle \overline{\mathcal{O}}_1 \ldots \overline{\mathcal{O}}_n \rangle = 0$$

  - alternative ring $[D, \overline{\mathcal{O}}\} = 0$ is deformed

Seiberg '03

- No chiral ring since $\overline{Q}$ is not a symmetry
- Gaugino condensate:
  perturbative and instanton corrections



Imaanpur '03

Summary

- String theory in graviphoton field
  - self–dual field: no gravitational backreaction
  - theory on the brane: deformed superspace
- Renomalization of $\mathcal{N} = \frac{1}{2}$ SYM
  - operators of dimensions 5 and 6
  - R charge constraint, no hermiticity
  - no new operators are generated
- Instanton solution
  - truncation of series in $C_{\alpha\beta}$
  - Laplace equations for prepotentials
- Metric on the moduli space
  - information metric vs L^2 metric
  - measure is not deformed
- Open problem: deformation of the chiral ring