Testing Dijkgraaf-Vafa conjecture at higher genus

Based on work with Albrecht Klemm (to appear)

Plan:
1. Introduction and Motivation
2. Reviews (Genus zero, one)
3. Dijkgraaf-Vafa conjecture at higher genus
Introduction

- The geometry:
  a hypersurface in $\mathbb{C}^4(v, w, x, y)$

\[ vw = w'(x)^2 + f(x) + y^2 \]

- $W(x)$: a polynomial of degree $n+1$.
- $f(x)$: a polynomial of degree $n-1$.
- $n$: number of complex structure moduli.

This is a local, non-compact Calabi-Yau.

- Matrix model:
  a bosonic matrix model, potential $W(x)$

- Consider $n$-cut solution in large $N$ limit
  $f(x)$ determines the filling fractions.
Motivations

1. Connections to SUSY gauge theory
   - $N=1$ superpotential
     \[ W_{\text{eff}} = \sum_i \left( N_i \frac{\partial F_0}{\partial s_i} + N_i s_i \left( -1 + \left( \frac{s_i}{\Lambda^0} \right)^2 \right) \right) \]
   - $N=2$ prepotential
   - Higher genus amplitudes = gravitational corrections

2. Connecting holomorphic anomaly equations with loop equations.

3. Lessons for large $N$ duality, open/closed string duality, $AdS/CFT$ correspondence ...
Review

- Consider a cubic potential
  \[ W(x) = \frac{1}{4} x^4 + \frac{1}{3} g x^3 \]

- \[ W(x)^2 + f(x) = 0 \] has 4 roots
  \[ (a_i^+, a_i^-) \equiv (x_1, x_2, x_3, x_4) \]

- Two complex structure moduli:
  \[ z_1 = \left( \frac{a_i^+ - a_i^-}{2} \right)^2, \quad z_2 = \left( \frac{a_i^+ + a_i^-}{2} \right)^2 \]

- Discriminant points:
  \[ \Delta_1 = 3 \Delta_2 \]
  \[ \Delta_2 = (x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_3 - x_4) \]
  \[ = \frac{1}{99} \left( 1 - 6g^2(x_1 + 2x_2) + g^4(9x_1^2 + 19x_2^2 + 9x_3^2) \right) \]
  \[ I = 1 - 2g^2(x_1 + 2x_2) \]

- \( I \) is not a discriminant point, but is a singularity of genus zero three point facts similarly as the orbifold point in quintic
Motivations

1. Connections to SUSY gauge theory
   - $N=1$ superpotential
     \[ W_{\text{eff}} = \sum_i (N_i \frac{\partial F_0}{\partial s_i} + N_i s_i (-1 + \frac{\partial s_i}{\partial b_i} + 1)) \]
   - $N=2$ prepotential
   - Higher genus amplitudes
     = gravitational corrections

2. Connecting holomorphic anomaly equations with loop equations

3. Lessons for large $N$ duality, open/closed string duality, AdS/CFT correspondence ...
• Fundamental periods

\[ S_i = \frac{1}{2\pi i} \int_{c_i}^a \omega, \quad \Pi_i = \frac{1}{2\pi i} \int_{c_i}^a \omega \]

\[ \omega = \sqrt{w(x)^2 + f(x)} \, dx \]

•

\[ S_1(z_1, z_2, g) = \frac{1}{4} \theta_1 - \frac{g^2}{8} \theta_1 (2z_1 + 3z_2) + O(g^7) \]

\[ S_2(z_1, z_2, g) = -S_1(z_2, z_1, g) \]

Similarly for \( \Pi_1 \), \( \Pi_2 \)

• Genus zero free energy \( F_0 \)

\[ \frac{\partial F_0}{\partial S_i} = \Pi_i (S_i) \]

• Three point Yukawa coupling

\[ C_{s_i s_j s_k} = \frac{\partial^3 F_0}{\partial s_i \partial s_j \partial s_k} \]

They are rational functions in \( S_i \) coordinate.
• The geometry on the complex structure moduli space of Calabi-Yau is a special Kahler geometry.

  Kahler potential : $K$

  Metric : $G_{ij} = \bar{\partial}_i \bar{\partial}_j K$

• Special geometry relation:

  $R_{i j k l}^k = G_{i j} \delta_{k l} + G_{k j} \delta_{i l} - C_{i l m} \bar{C}^{k m}_{j}$

• Simplifications for local geometry in holomorphic limit $\bar{\phi}_i \to 0$:

  1. Kahler potential $K$ is a constant.
  2. $G_{i k} \bar{\phi}_i = C_{k i}$ are constants in $\bar{\phi}_i$ coordinates.

So in $\bar{\phi}_i$ coordinates:

\[
\begin{align*}
G_{i \bar{j} \bar{i}} &= \frac{\partial G_{i k}}{\partial \bar{\phi}_i} C_k, \\
\Gamma^{\bar{i}}_{\bar{j} \bar{k}} &= -\frac{\partial \bar{\phi}_i}{\partial \bar{\phi}_j} \frac{\partial^2 S}{\partial \bar{\phi}_i \partial \bar{\phi}_j \partial \bar{\phi}_k}
\end{align*}
\]
Holomorphic anomaly equation

\[ \partial_i \bar{\partial}_j F^{(n)} = \frac{1}{2} \epsilon_{i j k l} \bar{e}_{j k l} e^{2k} \bar{G}^{k l} G_{i}^{k l} - \left( \frac{X}{2g} - 1 \right) G_{i}^{k l} \]

\[ \bar{\partial}_j F^{(g)} = \frac{1}{2} \epsilon_{j k l} e^{2k} \bar{G}^{k l} G_{i}^{k l} \left( D_{j} D_{k} F^{(g-1)} + \sum_{r=1}^{g-1} D_{j} F^{(r)} D_{k} F^{(g-r)} \right), \quad g \geq 1 \]

First term:

Second term:

- Holomorphic anomaly equation can be integrated with an integration constant: holomorphic ambiguity
• Matrix model

\[ Z = e^F = \frac{1}{\text{Vol}(U(N))} \int d\Phi e^{-w(\Phi)} \]

\[ w(\Phi) = \frac{1}{2} \phi^2 + \frac{9}{3} \phi^3, \quad \phi \text{ is a } N \times N \text{ matrix.} \]

• Two critical points \( w(x) = 0 \Rightarrow x = 0, -\frac{1}{9} \)

• Consider large \( N \) limit, two-cut solution, put \( N_1 \) eigenvalues at \( x = 0 \), \( N_2 \) eigenvalues at \( x = -\frac{1}{9} \), \( (N_1 + N_2 = N) \). Identify \( N_i = S_i \).

• Loop equations: (Ambjorn, Chekhov, Kristjansen, Makeenko, A. Klemm)

Genus zero: saddle point approximation. So far some cases have been solved

1. Multi-cut genus one
2. One-cut genus two

But two-cut genus two solution is not known.
Method: perturbative calculations

\[ Z = e^F \sim \int D\phi_1 D\phi_2 e^{-w_1(\phi_1) - w_2(\phi_2) - w(\phi_1, \phi_2)} \]

\[ w_1(\phi_1) = \text{tr}(\frac{1}{2} \phi_1^2 + \frac{1}{3} g \phi_1^3) \]

\[ w_2(\phi_2) = -\text{tr}(\frac{1}{2} \phi_2^2 - \frac{9}{3} \phi_2^3) \]

\[ w(\phi_1, \phi_2) = 2 \sum_{k=1}^{\infty} \frac{1}{k!} 9^k 2^k \sum_{p=0}^{k} (-1)^p \binom{k}{p} \text{tr}(\phi_1^p) \text{tr}(\phi_2^k) \]

\( \phi_i \) are \( N_i \times N_i \) matrices.
The model is perturbatively well defined.

Non-perturbative part:

\[ F_{\text{np}} = \frac{N_1^2}{2} \log(N_1) + \frac{N_2^2}{2} \log(N_2) - \frac{3}{4} (N_1^2 + N_2^2) \]

\[ -\frac{1}{12} \log (N_1 N_2) + 2 q'(\phi) \sum_{g=0}^{\infty} \frac{B_{2g}}{4g(9-1)} \left( \frac{1}{N_1^{2g+1}} + \frac{1}{N_2^{2g+1}} \right) \]

\[ F_{\text{np}} \] matches the universal leading behavior near conformal point.
Genus one case

(Klemm, Marino, Theisen, Dijkgraaf, Sin, Kovics, Temurhan)

- Integrate the genus one holomorphic anomaly equation

\[ 2 \bar{e} \bar{e} F'' = \frac{1}{2} C_{ikl} \bar{e} \bar{e} F_{ikl} C^{2k} C^{kh} G^{l \bar{h}} \]

\[ \Rightarrow F'' = \frac{1}{2} \log(\det(G_{ij})) f(\theta) \]

- Ansatz for holomorphic ambiguity

\[ f(\theta) = \Delta^{k_1} \Delta_{k_2}^{k_3} I_k \]

- Comparing with matrix model to fix three coefficients.
Higher genus

- Propagator \( \overline{\partial}^* S_k^i = \overline{\partial}^* S_k^i \)
- Integrate and use special geometry relation
  \[ S_k^i C_{ihe} = S_k^i \partial_e K + S_k^i \partial_e K + T_i^h e + f_i^h \]
- \( f_i^h \) are ambiguous integration constant, meromorphic rational functions with poles at discriminant points and special points of local geometry.

- \( \frac{1}{2} n^2(n+1) \) equations
  \( \frac{1}{2} n(n+1) \) propagators

  Impose constraints on \( f_k^i \) for \( n > 1 \)

- Another constrain
  \[ \overline{\partial} (\partial_i F^\omega) - \frac{1}{2} S_k^i c_{ijk} = 0 \]
  \( i \neq j \)

  \[ \partial_i F^\omega = \frac{1}{2} S_k^i c_{ijk} + \partial_i \psi \partial_\omega \log(\omega) \]
• Choose $e_i F^{(a)} = \frac{1}{2} \delta^{ij} C_{ijk} + e_i (\frac{1}{15 \log(\alpha)} + \frac{3}{15 \log(\alpha)})$

• Integrate genus two holomorphic anomaly equation

$$F^{(a)} = \frac{1}{8} \delta^{ij} \delta^{kl} F^{(0)}_{ijkl} + \frac{1}{2} \delta^{ij} F^{(0)}_{ij}$$

$$- \frac{1}{2} \delta^{ij} \delta^{kl} F^{(0)}_{ijkl} + \frac{1}{2} \delta^{ij} F^{(0)}_{ij}$$

$$+ \frac{1}{2} \delta^{ij} \delta^{kl} S_{mn} F^{(0)}_{imn} F^{(0)}_{km} + \frac{1}{8} \delta^{ij} \delta^{kl} S_{mn} F^{(0)}_{imn} F^{(0)}_{km}$$

• $a \equiv \frac{1}{15} \implies F^{(a)}$ has the leading behavior of

$$F^{(a)} \sim - \frac{1}{240} \left( \frac{1}{N_1^2 + N_2^2} \right)$$

$$= \frac{B_4}{2 \log(9-1)} \left( \frac{1}{N_1^2 + N_2^2} \right)$$
• Fix the holomorphic ambiguities

\[ f'_{10} = - \left( 6 - (4981 + 4880) \right) g^4 + \left( 63 \beta _1^2 + 219 \beta _2^2 + 
126 \beta _3^2 \right) g^6 + \left( 210 \beta _1^2 + 304 \beta _2^2 + 292 \beta _3^2 + 108 \right) g^8 / \left( 20 \beta _1^2 \Delta _2^1 \right) \]

\[ f'_{12} = \ldots \]

\[ f^{(0)} = - \left( 1253 - 10503 (\beta_1 + \beta_2) \right) g^2 + 27 (108 \beta_1^2 
+ 950 \beta_2^2 + 108 \beta_3^2) g^4 - 26865 (\beta_1 + \beta_2) (\beta_2 + \beta_3) g^6 / \left( 9000 g^4 \Delta_2^1 \right) \]

• We test the result to 8th order.

• Topological strings give an exact answer for genus two two-cut matrix model.
Some lessons

- Develop technique in both topological strings and matrix models.
- Confirm ideas of $C$-deform
  (Dijkgraaf, Gross, Ooguri, Vafa, Zamora, hep-th/0302109, 0303063, 0310061)
- Holomorphic anomaly equation is universal, but holomorphic ambiguity can be fixed
  by specific models.
  (Comparing with topological strings on another geometry \(O(-2, -2) \to \mathbb{P}^1 \times \mathbb{P}^1\)
  considered by Klecco, Marinos, Thellen, hep-th/0211216)