EXACT COUNTING
OF 4D BPS BLACK HOLES

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PLAN OF TALK

• 4D/5D Connection

• 1/4 BPS BHs
  in $\mathcal{N}=4$ string theory

• 1/8 BPS BHs
  in $\mathcal{N}=8$ string theory

• D4–D0 system on the quintic
  - gauge theory (quiver QM)
  - geometry

• Outlook
THE 4D/5D CONNECTION

$\mathbb{R}^{1,4} \xrightarrow{\text{KK}} \mathbb{R}^{1,3} \times S^1$

A 4D BH with 1 unit of KK monopole charge?

$\mathbb{R}^4 \xrightarrow{\text{Spinning}} \text{5D Taub-NUT BH}$

$R_{TN} \to \infty$

spinning 5D BH

BMPV
Type IIA string theory on a CY

BPS BH $\leftrightarrow$ D0 - D2 - D4 - D6 system

charge: $(\varphi_0, \varphi_A; P^A, P^0)$

D6 wrapped on CY
$\rightarrow$ KK monopole charge
($M$-theory on CY $\times S^1$)

CONSIDER D0 - D2 - D6 with a single D6-brane,

PROPOSAL:

$Z_{4D}(p^0=1, \varphi_0, \varphi_A) = Z_{5D}(\varphi_A, J_L = \frac{\varphi_0}{2})$

Subtlety: 4D index $\text{Tr}_{\text{BPS}} (-)^{2J^3}$

5D index $\text{Tr}_{\text{BPS}} (-)^{2J_L^3 + 2J_R^3}$

$J^3 \leftrightarrow J_L^3$
$\frac{\varphi_0}{2} \leftrightarrow J_L^3$

$Z_{5D} = (-)^{\varphi_0} Z_{4D}$
CLASSICAL ENTROPY

$D_0-D_2-D_6$

$$S_{4D} = 2\pi \sqrt{p^0 Q^3 - \frac{1}{4} (p^0 q_0)^2},$$

$$Q^3 \equiv (D_{ABC} y^A y^B y^C)^2,$$

$$q_A = 3D_{ABC} y^B y^C.$$

\[ p^0 = 1 \]

$$S_{4D} = 2\pi \sqrt{Q^3 - \frac{1}{4} q_0^2}$$

BMPV

$$S_{5D} = 2\pi \sqrt{Q^3 - J^2}$$
$N=4$ String theory
- IIA on $K3 \times T^2$

A 1/4-BPS BH

$D_0 - D_2 - D_6$ $(p^a = 1)$  \[\text{lift to 5D} \rightarrow \text{M2 on } K3 \times T^2 \text{ w/ } J_L\]

IIA: $D_2$ on $K3$
+ $F_1$ on $S^1$
+ $J_L$

compactify on $S^1 < T^2$

IIB: $D_3$ on $K3 \times S^1$
+ momentum along $S^1$
+ $J_L$

\[\text{T-duality} \rightarrow \text{U-duality}\]

IIB: $D_1 - D_5$ on $K3 \times S^1$
+ momentum along $S^1$
+ $J_L$
D1-D5 on $K3 \times S^1$

$\rightarrow \text{Sym}_{q, q_5}^q(K3) \quad \text{CFT}$

$K3$

elliptic genus:

$$X_{N=q, q_5} (y, \varphi) = \text{Tr} \left( (-)^F y^{F_L} \varphi^{Q_0} \right)$$

$\text{4D/5D Connection}$

$$Z_{4D} = Z_0 \cdot Z_{\text{D1-D5}}$$

partition function of $D5$ on $K3 \times S^1$

winding modes of a heterotic string
\[ Z_{\text{D1-D5}} = \sum d_{5D}(L, N, J_L) e^{2\pi i (L_0 \rho + N \sigma + 2J_L \nu)} \]

\[ = \prod_{k>0, l>0} \prod_{m \in \mathbb{Z}} (1 - e^{2\pi i (k \rho + l \sigma + m \nu)})^{c(4kl-m^2)} \]

Dijkgraaf, Moore, Verlinde, Verlinde

\[ c(4kl-m^2) : \text{coefficients of elliptic genus of a single K3} \]

\[ Z_0 = (e^{\pi i \nu} - e^{-\pi i \nu})^{-2} e^{-2\pi i \rho} \]

\[ \times \prod_{n \geq 1} (1 - e^{2\pi i (n \rho + \nu)})^{-2} (1 - e^{2\pi i (n \rho - \nu)})^{-2} \]

\[ \times (1 - e^{2\pi i n \rho})^{-20} \]

Antoniadis, Gava, Nariai, Taylor

Weight 10 modular form of Sp(2, \mathbb{Z})

Dijkgraaf, Verlinde, Verlinde

\[ Z_0 \cdot Z_{\text{D1-D5}} = \frac{e^{2\pi i \sigma}}{\Phi(\rho, \sigma, \nu)} \]

\[ \text{shift of } Q_1 Q_5 \text{ by due to anomalous charge} \]
$N=4$ string theory has U-duality group

$$SL(2,\mathbb{Z}) \times O(6,22;\mathbb{Z})$$

Charge vectors: (ignoring F1 and NS5 charges)

$\mathbf{q}_{e} = (q_{0}, q_{1}, ..., q_{22}, p_{23}^{\uparrow})$

$D0 \ D2 \text{ on } K3 \quad D4 \text{ on } K3$

$\mathbf{q}_{m} = (p_{0}, p_{1}, ..., p_{22}, q_{23}^{\uparrow})$

$D6 \ D4 \text{ on } \alpha \times T^{2} \quad D2 \text{ on } T^{2}$

BH degeneracy

$d_{4D}(\mathbf{q}_{e}, \mathbf{q}_{m}) = d(\mathbf{q}_{e}^{2}, \mathbf{q}_{m}^{2}, \mathbf{q}_{e} \cdot \mathbf{q}_{m})$

$L_{0} \quad Q_{1}Q_{5} \quad J_{L}$

$Z_{4D} = \sum_{k,l,m} d(k, l, m) \ e^{2\pi i(kp + lp + mv)}$

$= \frac{1}{\Phi(q, \sigma, \nu)}$
$N = 8$ String theory
- IIA on $T^6$

Similar chain of dualities as in $N=4$ case

$\Rightarrow Z_{4D} = Z_{D1-D5}$

$= Z_{Sym^2 \mathbb{Q}_5(T^4)}$

- $Z_{D1-D5}$ is given by a modified elliptic genus
  $\text{Tr} \ (-)^F (J^3)^2 y^2 J^3 q^L \bar{q}^L$

  Maldacena, Moore, Strominger

Correspondingly

$Z_{4D} = \text{Tr}_{BPS} (J^3)^2 (-)^2 J^3$

- $Z_0$ is absent - part. fn. of $D5$ on $T^4 \times S$ trivial

- Need to assume coprime set of
  $(N=Q_1, Q_5, L_0, J_L)$ to get $Z_{4D}$
  that is consistent with $U$-duality $E_{7,7}$

$Z_{4D} = \sum d(J) q^J$

Cremmer - Julia invariant
\[ Z_{4D} = \eta(q^4)^{-6} \sum_{m \in \mathbb{Z}} q^{m^2} \]

**A 4D DERIVATION?**

- degeneracy depends only on \( J \)
- D4-D0 system
  \[ J = 4q_0 D = 4q_0 p^1 p^2 p^3 \]

**CONSIDER** \( p^1 = p^2 = p^3 = 1 \)

\[ J = 4q_0 \]

- \( J \equiv 0 \) or \(-1 \mod 4 \)

  The other case can be obtained by turning on D2-brane charge

- Do partition function is then

\[ Z(q) = \sum_{n} d(J = 4n) q^n = \eta(q)^{-6} \sum_{m \in \mathbb{Z}} q^{(m+\frac{1}{2})^2} \]

D4 wrapped on \((1,1,1)\) cycle \( \mathbb{P} \perp T^6 \)
\[ P \subset T^6 \quad \text{cplx submanifold for} \]
\[ \text{generic (tilted) } T^6 \quad \text{(algebraic)} \]

\[
\begin{array}{ccc}
1 & & \\
3 & 3 & \\
3 & 10 & 3 \\
3 & 3 & \\
1 & & \\
\end{array}
\]

\[ \chi(p) = 6 \]
\[ \sigma(p) = -2 \]
\[ b_1(p) = 6 \]
\[ b_2(p) = 16 = b_2(T^6) + 1 \]

- \( P \) has an extra \( 2 \)-cycle \( \gamma \) not induced from \( T^6 \)
- Can turn on gauge field flux without induced \( D2 \)-brane charge
  but with induced \( D0 \)-brane charge

\[ \alpha_1, \alpha_2, \alpha_3 \quad \text{2-cycles induced from} \ T^6 \]
\[ \gamma \cdot \alpha_i = 1, \quad \gamma \cdot \gamma = 1 \]
\[ \beta = 2\gamma - \Sigma \alpha_i, \quad \beta \cdot \alpha_i = 0, \quad \beta \cdot \beta = -2 \]

- Freed–Witten anomaly:

\[ F = \frac{c_1(p)}{2} + \text{integral} \]
\[ c_1(p) = -\Sigma \alpha_i \]
Allowed fluxes that do not induce D2-charge:

\[ F = (m + \frac{1}{2})^2, \quad m \in \mathbb{Z} \]

Induce D0-brane charge

\[ \Delta \Phi_0 = -\int \frac{F^2}{2} + \frac{c_2(p)}{24} \]

\[ = (m + \frac{1}{2})^2 - \frac{1}{4} \]

D4-D0 bound state:

DO either dissolve into F

or be bound to D4 as instantons

\[ \Rightarrow Z = \eta(\xi)^{-6} \sum_{m \in \mathbb{Z}} e^{\frac{1}{8}(m + \frac{1}{2})^2} \]

agree with 5D!
$N=2$ string theory

- IIA on CY $X =$ quintic

Consider $N$ D4-branes wrapped on $P_1$ - minimal 4-cycle (hyperplane section)

Can we describe all the moduli of the wrapped D4-brane in terms of the world-volume quantum mechanics?

- Dimensional reduction of $N=1$ $U(N)$ gauge theory + 4 adjoint matter
A PUZZLE

Geometry: can deform N D4 wrapped on P_i to a single D4 wrapped on P_n. \([P_n] = N[P_i]\)

A degree N divisor in \(X\) has

\[\sim N^3\] moduli

\[
\# \text{ moduli} = \dim H^0(L_{P_n}) - 1
= \int_X ch(L_{P_n}) \wedge Td(X) - 1
= \frac{5N^3 + 25N}{6} - 1
\]

\(\sim\) a U(1) gauge theory with

\[\sim N^3\] matter fields

Impossible to obtain from (the Higgs branch) a gauge theory with \(\sim N^2\) degrees of freedom. ?
A CLUE: Induced charges

D-brane coupling to RR fields

\[ \int \text{ch}(E) \wedge \sqrt{\frac{A(TP)}{A(NP)}} \wedge \sum_p C_p \]

D4-brane wrapped on P with gauge flux F:

D2-charge

\[ q_2 = F \cdot J \]

D0-charge

\[ q_0 = -\int_P \frac{F^2}{2} + \frac{C_2(TP)}{24} \]

Freed-Witten anomaly:

\[ F = \frac{C_1(\text{TP})}{2} + \text{integral} \]

\[ = -\frac{N}{2} [P_1] + \text{integral} \]

\[ \uparrow \quad [P_1] = J \]

\[ P_N \text{ is } \begin{cases} \text{spin}, & N \text{ even} \\ \text{not spin}, & N \text{ odd} \end{cases} \]
A D4-brane wrapped on $P_1$: can choose $F = \frac{J}{2}$

Induce: $q_2 = \frac{5}{2}$
$q_0 = -\frac{35}{12}$

When $N$ $P_1$-wrapped D4-branes join together into $P_N$, must have

$$F = \frac{J}{2} + F^-$$

so that the induced D2-charge is conserved.

$F$ must be (half) integral for $N$ even (odd).

$F \cdot F$ must give the right D0-charge.

Such $F$ may not exist for generic degree $N$ surface in $X$!
An integral $(1, 1)$ harmonic 2-form $\leftrightarrow$ a divisor (linear combination of curves) in $P_n$.

If not proportional to $J$:
Need holomorphic curve $C \subset P_n$ that are not homologous to multiples of $J$, i.e. not complete intersection in $X$.

**THE SIMPLEST EXAMPLE**

$x y - z w = 0$

contains a curve defined by $x = z = 0$

not a complete intersection.
A CLASS OF EXAMPLES

\( \Phi_A \) - N\times N cplx matrices
\( A = 1, \ldots, 5 \)

\( P_N: \det (\Phi_A z^A) = 0 \)
- degree N surface in \( X \)
  \( (\subset \mathbb{P}^4) \)

Consider curve \( C \) defined by

\( \Phi_{i} = 0, \quad i = 1, \ldots, N \)

\[ \text{minors of } \Phi = \Phi_A z^A \]

CLAIM:

\[ F = C - \frac{N}{2} J \]

is the needed flux on \( P_N \)!
NEED TO COMPUTE
\[ C \cdot J = ? \quad C \cdot C = ? \]

\[ Q_5(z^A) = 0 \quad \text{quintic equation} \]
\[ J: \quad h_i z^A = 0 \quad \text{hyperplane section} \]
\[ \sum_{i=2}^{N} a_i \varPhi_{ij}(z) = 0, \quad j=1, \ldots, N \]

\[ [z^A] \in \mathbb{P}^4, \quad [a_i] \in \mathbb{P}^{N-2} \]

\[ \rightarrow \text{counting zeros of a section of a vector bundle} \]
\[ V \quad \text{rk} V = N+2 \]
\[ \downarrow \]
\[ \mathbb{P}^4 \times \mathbb{P}^{N-2} \]
\[ \mathcal{X}_Y \quad \text{hyperplane} \]

\[ V = L^5_x \oplus L_x \oplus (L_x \oplus L_y)^{\otimes N} \]
\[ \uparrow \quad \uparrow \]
\[ Q_5 \quad J \]
\[ C \cdot J = \int_{\mathbb{R}^4 \times \mathbb{R}^{N-2}} e(V) \]

\[ = \int_{\mathbb{R}^4 \times \mathbb{R}^{N-2}} 5x \cdot x \cdot (x+y)^N \]

\[ = 5 \binom{N}{2} \]

Similarly, can compute

\[ C \cdot C = 5 \binom{N}{3} \]

Straightforward to check:

\[ F = C - \frac{N}{2} J \] gives induced charges

\[ \varphi_2 = \frac{5}{2} N \]

\[ \varphi_0 = -\frac{35}{12} N \text{ (} N^3 \text{ terms cancel) } \]

- \( N \) times the charges of \( P_1 \)-wrapped D4-brane with \( F = J/2 \)!

AGREEMENT
GAUGE THEORY INTERPRETATION

N D4 wrapped on P₁

⇒ (dim'l reduction of)

N=1 U(N) SYM + 4 adj matter

\sim \Phi_A \quad A=1,\ldots,5

projective variable

moduli space

\sim \text{ locally } \mathbb{C}^{4N^2} \big/ \text{GL}(N,\mathbb{C})

→ 3N²+1 dimensional

agree w/ # moduli of the surface

\text{P}_N: \quad \det (\Phi_A Z^A) = 0.

5N² - N² - (N² - 1) = 3N² + 1
GEOMETRY OF D4-BRANE?
- D0-brane probe

Superpotential: \( W = \phi_04 \Phi_A \phi_{40} z^A \)

D0 coinciding with D4
\[ \uparrow \]
Massless 0-4 strings
\[ \uparrow \]
\[ \det(\Phi_A z^A) = 0 \rightarrow P_N! \]
COMMENTS

- Can generalize to $N$ D4-branes wrapped on $P_d$ ($d \geq 1$) straightforward
- Can generalize to the geometry of joining D4-branes wrapped on various different cycles, quiver gauge theory dual?
- Can generalize to D4-branes on CYs as complete intersections in toric varieties coming up
- The geometry of $P_n$ naturally arise in GLoM of open strings

Hellerman, Kachru, Lawrence, McGreevy

... work in progress...
OTHER LESSONS

- D4-D0 bound states consist of
  - Do's dissolving into F-
  - Do's as instantons (point-like)
  An exact counting of $N=2$ BH?

- D4-D0 quiver QM
  - Integrate out 0-4 strings
  flow to SCQM? Denef

- Counting large BHs (exactly)
  from quiver moduli space?
  - D6-D6 system
  - Superpotential?