NS-NS fluxes in Hitchin's generalized geometry

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Introduction

Generalized (complex) geometry, developed by Hitchin and Gualtieri has emerged as a useful framework for describing **new string compactifications**.

It naturally includes a large class of vacua known as **generalized Calabi-Yau manifolds**. [Hitchin; Gualtieri; Grana, Minasian, Petrini, Tomasiello; Grana, Louis, Waldram; Benmachichi, Grimm]

It also gives a more elegant description of so-called **non-geometric** spaces or **T-folds**. [Hellerman, McGreevy, Williams; Dabholkar, Hull; Kachru, Schulz Trpathy; Hull; Hull, Reid-Edwards; Grange, Schaker-Nameki]

One of the nice feature of generalized geometry is that it is naturally **covariant** under **T-duality**.

Since the metric $g_{\mu\nu}$ and B-field mix under T-duality, they typically never appear separately in generalized geometry, but only in the combination

$$\mathsf{G} = \begin{pmatrix} -g^{-1}B & g^{-1} \\ g - Bg^{-1}B & Bg^{-1} \end{pmatrix}.$$

This creates a puzzle: If g and B are combined into G, what should we do with $H = dB$? In particular, we would like to know the generalized analogue of

$$\mathsf{Flux} = \int_{\Sigma} H$$

Since generalized geometry is **covariant** under T-duality, the generalized analogue of H-flux should capture its various T-duals.

$$H\text{-flux} \xrightarrow{\mathsf{T}} \text{Geometric flux} \xrightarrow{\mathsf{T}} \text{Non-geometric } Q\text{-flux}$$

Proposal: The generalized version of H-flux is given by

$$\mathsf{H}(\mathsf{V}_1, \mathsf{V}_2, \mathsf{V}_3) = -\,\mathsf{Nij}(\widetilde{\mathsf{V}}_1, \widetilde{\mathsf{V}}_2, \widetilde{\mathsf{V}}_3)\ ,$$

where $\mathsf{Nij} \equiv$ Nijenhuis operator and $\widetilde{\mathsf{V}} = \mathsf{G}\mathsf{V}$.

We show this in two ways:

First:

$$\text{Flux} = \int_{\Sigma} \mathsf{H}$$

Second:

$$\mathsf{H} = \text{Torsion of } \mathsf{D}^{I}$$

Review of generalized geometry

To understand the **physical motivation** for generalized geometry, it is useful to recall the basics of **T-duality on the world-sheet**.

At the classical level, **T-duality** is **Poincaré** duality [Buscher]:

Start with

$$S = \frac{1}{2}\int g_{\mu\nu}\, dX^\mu \wedge *dX^\nu + B_{\mu\nu}\, dX^\mu \wedge dX^\nu \ . \qquad (\spadesuit)$$

If g and B don't depend on X^μ, we can replace $dX^\mu \to V^\mu$, provided we add a Lagrange multiplier term.

$$S = \frac{1}{2}\int g_{\mu\nu}\, V^\mu \wedge *V^\nu + B_{\mu\nu}\, V^\mu \wedge V^\nu + 2\hat{X}_\mu dV^\mu \ . \qquad (\clubsuit)$$

Integrating out $\hat{X}_\mu$ gives back ($\spadesuit$). Integrating out V^μ gives

$$d\hat{X}_\mu = g_{\mu\nu} * V^\mu + B_{\mu\nu} V^\mu \ .$$

Solving this equation for V^μ and substituting it into ($\clubsuit$) gives the **T-dual action** to ($\spadesuit$). We also learn the **on-shell** relationship between $\hat{X}_\mu$ and X^μ

$$d\hat{X}_\mu = g_{\mu\nu} * dX^\mu + B_{\mu\nu}\, dX^\mu \ .$$

The basic idea behind **generalized geometry** is that if we combine dX^μ and $d\hat{X}_\mu$ into a vector [Duff],

$$\begin{pmatrix} dX^\mu \\ d\hat{X}_\mu \end{pmatrix} , \qquad (\bigstar)$$

T-duality acts in a very simple way by exchanging elements of the top with elements of the bottom.

Loosely speaking, a **generalized vector** is an object [Hitchin; Gualtieri],

$$\mathsf{V} = \begin{pmatrix} V^\mu \\ \omega_\mu \end{pmatrix} , \qquad V \in T , \qquad \omega \in T^* ,$$

which transforms in the same way as $(\bigstar)$. Note that $\mathsf{V} \in \mathsf{E} = T \oplus T^*$.

We will also write elements of E as a formal sum of a vector and a form:

$$\mathsf{V} = V + \omega$$

as is common in the generalized literature.

Symmetries of E

It is useful to understand explicitly how various symmetries act on $\mathsf{E} = T \oplus T^*$

We begin with T-duality.

By construction, T-duality acts in a simple way. For example, if we have two dimensions and T-dualize along the 1-direction,

$$\mathsf{V} = \begin{pmatrix} V^1 \\ V^2 \\ \omega_1 \\ \omega_2 \end{pmatrix} \xrightarrow{\mathsf{T}_1} \begin{pmatrix} & & 1 & \\ & 1 & & \\ 1 & & & \\ & & & 1 \end{pmatrix} \begin{pmatrix} V^1 \\ V^2 \\ \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} \omega_1 \\ V^2 \\ V^1 \\ \omega_2 \end{pmatrix} .$$

T-duality will only be an allowed transformation when the **direction we are T-dualizing along is a $U(1)$ isometry.**

Whenever we speak of an object transforming covariantly under T-duality, we will always mean this restricted sense.

Diffeomorphisms act in the natural way on the vector and form indices.

Explicitly, if we transform coordinates from X^μ to $X^{\mu'}$ and define $M^{\mu'}{}_\mu = \partial X^{\mu'}/\partial X^\mu$ then V transforms as

$$\mathsf{V} \to \begin{pmatrix} (M^{-1})^{\mathsf{T}} & 0 \\ 0 & M \end{pmatrix} \mathsf{V} \ .$$

Gauge transformations, $B \to B + d\lambda$ also act in a simple way. Recall the relation,

$$d\hat{X}_\mu = g_{\mu\nu} * dX^\mu + B_{\mu\nu}\, dX^\mu \ .$$

Thus, under $B \to B + d\lambda$,

$$\begin{pmatrix} dX^\mu \\ d\hat{X}_\mu \end{pmatrix} \to \begin{pmatrix} dX^\mu \\ d\hat{X}_\mu + (d\lambda)_{\mu\nu}\, dX^\mu \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ d\lambda & 1 \end{pmatrix} \begin{pmatrix} dX^\mu \\ d\hat{X}_\mu \end{pmatrix}$$

For the generalized vector, we act in the same way:

$$\mathsf{V} \to \begin{pmatrix} 1 & 0 \\ d\lambda & 1 \end{pmatrix} \mathsf{V} \ .$$

Such a gauge transformation is referred to in the generalized literature as a **B-transformation**.

Using the "form plus vector" notation, it often denoted,

$$e^{\delta B}(V+\omega)=V+\omega+i_V\delta B\ ,$$

where $\delta B = d\lambda$, and we consider δB to be acting from the left by contracting indices with vectors.

Note that, as is standard, for a form $\rho_{\mu_1\ldots\mu_n}$, we define

$$(i_V\rho)_{\mu_2\ldots\mu_n}=V^{\mu_1}\rho_{\mu_1\ldots\mu_n}$$

The canonical metric

The diffeomorphisms, B-transformations and T-duality transformations are all **symmetries** of the **canonical inner product** given by

$$\langle \mathsf{V}_1, \mathsf{V}_2 \rangle = \langle V_1 + \omega_1, V_2 + \omega_2 \rangle = \omega_1(V_2) + \omega_2(V_1) \ ,$$

where $\omega(V) = V^\mu \omega_\mu$.

This metric has signature (d, d) and is thus invariant under $SO(d, d)$.

Note that the full local $SO(d, d)$ symmetry is only partially realized in generalized geometry because of the restriction that **B-transformations** be exact and that one can only **T-dualize** along $U(1)$-isometries.

The splitting

So far, we have been ignoring the fact that the elements of our generalized vector,

$$\begin{pmatrix} dX^\mu \\ d\hat{X}_\mu \end{pmatrix} ,$$

are not indepent but are related by

$$d\hat{X}_\mu = g_{\mu\nu} * dX^\mu + B_{\mu\nu}\, dX^\mu \ .$$

Note that depending on whether we are looking at **right-movers** or **left-movers**, this equation reduces to

$$d\hat{X}_\mu = \pm g_{\mu\nu}\, dX^\mu + B_{\mu\nu}\, dX^\mu \ .$$

This condition naturally defines two **subspaces**, [Hitchin; Gualtieri]

$$\mathsf{C}^\pm \equiv \mathsf{span} \left\{ \begin{pmatrix} V^\mu \\ \pm g_{\mu\nu} V^\nu + B_{\mu\nu} V^\nu \end{pmatrix} \middle| \ V^\mu \in T \right\} \ ,$$

which can be thought of as the **right/left-moving** halves of the bundle $\mathsf{E} = T \oplus T^*$.

Conveniently, these two spaces are orthogonal under the **canonical inner-product** and satisfy $\mathsf{C}^+ \oplus \mathsf{C}^- = \mathsf{E}$. They therefore define a **splitting** of E.

This **splitting** is conveniently **encoded by a matrix** G with eigenvalues ± 1 for elements in $\mathsf{C}^\pm$. [Duff; Tseytlin; Hitchin; Gualtieri]

$$\mathsf{G} = \begin{pmatrix} -g^{-1}B & g^{-1} \\ g - Bg^{-1}B & Bg^{-1} \end{pmatrix}.$$

Note that $\mathsf{G}^2 = 1$, which follows from the fact that its eigenvalues are ± 1.

Heuristically, G should be thought of as the analogue of the Hodge star, $*$, on the world sheet.

In generalized geometry, we never speak of the metric and B-field separately, it is only the **combination** G which enters the story.

If $\mathsf{\Lambda}$ is some combination of diffeomorphisms, B-transformations and T-dualities, G transforms as $\mathsf{G} \rightarrow \mathsf{\Lambda G \Lambda}^{-1}$.

Thus if V is a generalized vector, than so is

$$\widetilde{V} = \mathsf{GV} \ .$$

Using G one can define a **positive-definite inner product** on E;

$$\mathsf{G(A, B)} = \langle \mathsf{A, GB} \rangle = \langle \mathsf{GA, B} \rangle \ .$$

This inner product often acts as the **generalized version** of the **metric** g.

The Courant-bracket

A basic object in generalized geometry is the **Courant-bracket**, [Hitchin; Gualtieri]

$$[V_1+\omega_1, V_2+\omega_2]_C = [V_1, V_2]_L + \mathcal{L}_{V_1}\omega_2 - \mathcal{L}_{V_2}\omega_1 - \frac{1}{2}(d(i_{V_1}\omega_2) - d(i_{V_2}\omega_1)) \ ,$$

where $[V_1, V_2]_L$ is the **Lie-bracket** of two vector fields and $\mathcal{L}_V = i_V d + d i_V$ is the **Lie-derivative**.

As with the other objects we have defined, the **Courant-bracket** is **covariant** under diffeomorphisms, B-transformations and T-duality;

$$[\mathsf{\Lambda}\mathsf{V}_1, \mathsf{\Lambda}\mathsf{V}_2]_C = \mathsf{\Lambda}[\mathsf{V}_1, \mathsf{V}_2]_C \ .$$

The **Courant-bracket** should be thought of as the **generalized version** of the **Lie-bracket** of two vector fields.

An interesting property of the Courant-bracket is that **it does not satisfy the Jacobi identity**. Rather, it has Jacobiator [Gualtieri]

$$[[\mathsf{V}_1, \mathsf{V}_2]_C, \mathsf{V}_3]_C + \mathsf{cyclic} = d\,\mathsf{Nij}(\mathsf{V}_1, \mathsf{V}_2, \mathsf{V}_3) \ ,$$

where the **Nijenhuis operator** is defined by [Gualtieri]

$$\mathsf{Nij}(\mathsf{V}_1, \mathsf{V}_2, \mathsf{V}_3) = \frac{1}{3}\langle [\mathsf{V}_1, \mathsf{V}_2]_C, \mathsf{V}_3\rangle + \mathsf{cyclic} \ .$$

The **Nijenhuis operator**, as have already seen, plays an important role in defining the generalized flux.

The generalized flux

Computing the flux associated with the 3-form $H = dB$ can be thought of as having three ingredients:

1. The **flux** H,

2. The **cycle** Σ we wish to integrate it over

3. The actual **integration**, $\int_\Sigma H$.

Lifting this computation to **generalized geometry** requires **modifying** each of these **notions**.

Generalized p-cycle

Let's begin by extending the notion of a p-cycle.

A **generalized p-cycle** will be given by **two ingredients**.

The **first ingredient** is just an ordinary p-dimensional manifold Σ, which is a **submanifold** of our spacetime manifold M.

Given such a Σ, we can try to **pull back** the bundle $\mathsf{E} = T_M \oplus T^*_M$ to Σ. There is a slight **subtlety** in doing this; in the presence of a nontrivial B-field, the bundle E is twisted by a gerbe.

However, since we can pull back the B-field to Σ, we can just put locally $\mathsf{E}_\Sigma = T_\Sigma \oplus T^*_\Sigma$, where it is understood that globally T^*_Σ is twisted by the pullback of the B-field.

We can also pull back the splitting of E into $\mathsf{C}^+ \oplus \mathsf{C}^-$. This is accomplished by pulling back g and B to Σ and then constructing the matrix G.

Our **second ingredient** will be set of **generalized vielbeins**

$$\mathsf{V}_i \in \mathsf{E}_\Sigma \ , \qquad\qquad i \in \{1, 2, \ldots, p\} \ ,$$

that satisfy the generalized version of the **standard property of vielbeins**,

$$\mathsf{G}(\mathsf{V}_i \, , \mathsf{V}_j) = \delta_{ij} \ .$$

The **purpose** of these vielbeins, as we will see later, is to tell us how to **pull back** generalized objects to Σ so that we can integrate them.

Such vielbeins will typically not exist globally. Thus, we also allow gauge rotations $\mathsf{V}_i \to O_i{}^j \mathsf{V}_j$ for $O \in SO(p)$ between patches of the space.

Together, the **ordinary cycle** Σ and the **colection of vielbeins** V_i will be called a **generalized** 3**-cycle** and will be denoted $\boldsymbol{\Sigma}$.

A measure for integration

In order to get some intuition for the role of the V_i, consider the case when the they have a **vanishing vector component**,

$$\mathsf{V}_i = \begin{pmatrix} 0 \\ \omega_i \end{pmatrix} , \qquad \omega_i \in T^*_\Sigma \ .$$

The property $\mathsf{G}(\mathsf{V}_i, \mathsf{V}_j) = \delta_{ij}$, using the explicit expression

$$\mathsf{G}(\mathsf{V}_i, \mathsf{V}_j) = \mathsf{V}_i^\mathsf{T} \begin{pmatrix} g - Bg^{-1}B & Bg^{-1} \\ -g^{-1}B & g^{-1} \end{pmatrix} \mathsf{V}_j \ ,$$

reduces to

$$\omega_{i\mu}\omega_{j\nu}g^{\mu\nu} = \delta_{ij} \ .$$

Thus, the $\omega_{i\mu}$ are **ordinary vielbeins**.

To define a measure, note that in the **“form plus vector”** notation, we have simply

$$\mathsf{V}_i = \omega_i \ .$$

Hence, we can simply wedge the V_i together,

$$\mathsf{V}_1 \wedge \mathsf{V}_2 \wedge \ldots \wedge \mathsf{V}_p = \omega_1 \wedge \omega_2 \wedge \ldots \wedge \omega_p \ .$$

If we have coordinates $\xi^{1,2,\ldots,p}$ on our surface Σ, this reduces to

$$\begin{aligned} &\omega_{1\mu}\omega_{2\nu}\ldots\omega_{p\rho}\, d\xi^\mu \wedge d\xi^\nu \wedge \ldots \wedge d\xi^\rho \\ &\qquad = \det_{i\mu}(\omega_{i\mu}) d\xi^1 \wedge d\xi^2 \wedge \ldots \wedge d\xi^p \\ &\qquad\qquad = \sqrt{g}\, d\xi^1 \wedge d\xi^2 \wedge \ldots \wedge d\xi^p \ , \end{aligned}$$

where $g = \det(g_{\mu\nu})$.

This gives us a suitable measure for **integrating a scalar**.

To define a measure for a general set of V_i, consider that under T-duality along, say, the ξ^1 direction,

$$d\xi^1 \rightarrow \frac{\partial}{\partial \xi^1} \ .$$

Thus an integration measure,

$$d\xi^1 \wedge d\xi^2 \wedge d\xi^3 \ ,$$

would *formally* become the somewhat absurd looking

$$\frac{\partial}{\partial \xi^1} \wedge d\xi^2 \wedge d\xi^3 \ .$$

This odd looking measure should be interpreted as telling us that **one of the directions we are integrating over is actually a coordinate on the T-dual circle**.

Indeed, it is convenient to make the *formal* replacement,

$$\frac{\partial}{\partial \xi^\mu} \to d\hat{\xi}_\mu \ ,$$

where $\hat{\xi}_\mu$ is the coordinate T-dual to ξ^μ. We can then write a vector field as

$$V^\mu \frac{\partial}{\partial \xi^\mu} \to V^\mu d\hat{\xi}_\mu \ .$$

Our measure now takes the more visually appealing form,

$$d\hat{\xi}_1 \wedge d\xi^2 \wedge d\xi^3 \ .$$

Intuitively, as the notation suggests, we should integrate over $\xi^{2,3}$ using standard integration while for the ξ^1 coordinate, we should integrate over its T-dual, $\hat{\xi}_1$.

Postponing, for the moment, the precise rules for doing this, we can write down the formal measure:

$$\mathsf{V}_1 \wedge \mathsf{V}_2 \wedge \ldots \wedge \mathsf{V}_p \ ,$$

Integration over the generalized measure

It is useful to start with the example of a **generalized 1-cycle** with a single coordinate, ξ^1, with period $\Delta\xi^1$.

In this case, we have a single vielbein V. Suppose that we take $\mathsf{V} = V$ where V is a one component vector. We would like to define

$$\int \mathsf{V} = \int V \ .$$

From $\mathsf{G}(\mathsf{V}, \mathsf{V}) = 1$ we learn that

$$(V^1)^2 g_{11} = 1 \ , \qquad \Longrightarrow \qquad V = \frac{1}{\sqrt{g_{11}}} \frac{\partial}{\partial \xi^1}$$

Assuming that g_{11} does not depend on ξ^1, we can write this as

$$V = \sqrt{\hat{g}_{11}}\, d\hat{\xi}_1 \ ,$$

where $\hat{g}_{11} = g_{11}^{-1}$ is the metric of the dual circle.

It is now clear how we can integrate over V. We put

$$\int V = \int \sqrt{\hat{g}_{11}}\, d\hat{\xi}_1 = \hat{L} \ ,$$

where $\hat{L}$ is the length of the dual circle.

Noting that

$$\hat{L} = \frac{1}{L} = \frac{1}{\sqrt{g_{11}}\Delta\xi^1} = \frac{\sqrt{\hat{g}_{11}}}{\Delta\xi^1}$$

we learn that

$$\int d\hat{\xi}_1 \equiv \frac{1}{\Delta\xi^1} \ ,$$

which is just what one would expect for the period of the dual circle.

This is the basic definition that will allow us to integrate over the generalized measure.

Note that it was important that our **integrand did not depend on the direction we were integrating over**. It would be very interesting if there were a natural definition of

$$\int f(\xi) d\hat{\xi} = ? \ ,$$

but it seems unlikely that one can make sense of such an object. Instead we will insist that whenever we have an integral of the form,

$$\int f(\xi^\mu)\, d\xi^1 \wedge \ldots \wedge d\xi^q \wedge d\hat{\xi}_{q+1} \wedge \ldots \wedge d\hat{\xi}_p \ , \qquad (\bigstar)$$

that $f(\xi^\mu)$ only depends on $\xi^{1,2,\ldots,p}$ and that $\xi^{p+1,\ldots,d}$ are periodically identified with period $\Delta\xi^i$. We can then repeatedly apply the formula,

$$\int d\hat{\xi}_i \equiv \frac{1}{\Delta\xi^i} \ ,$$

to yield

$$\left(\prod_{i=q+1}^{d} \frac{1}{\Delta\xi^i}\right) \int f(\xi^\mu)\, d\xi^1 \wedge \ldots \wedge d\xi^q \ .$$

This reduces the rather mysterious looking integral $(\bigstar)$ to an ordinary integral.

The fiber condition

To complete our definition of integration, we need to impose that the **integral over the generalized measure** can always be **reduced** to an **ordinary integral** by repeated application of the rule,

$$\int d\hat{\xi}_i \equiv \frac{1}{\Delta\xi^i} \ .$$

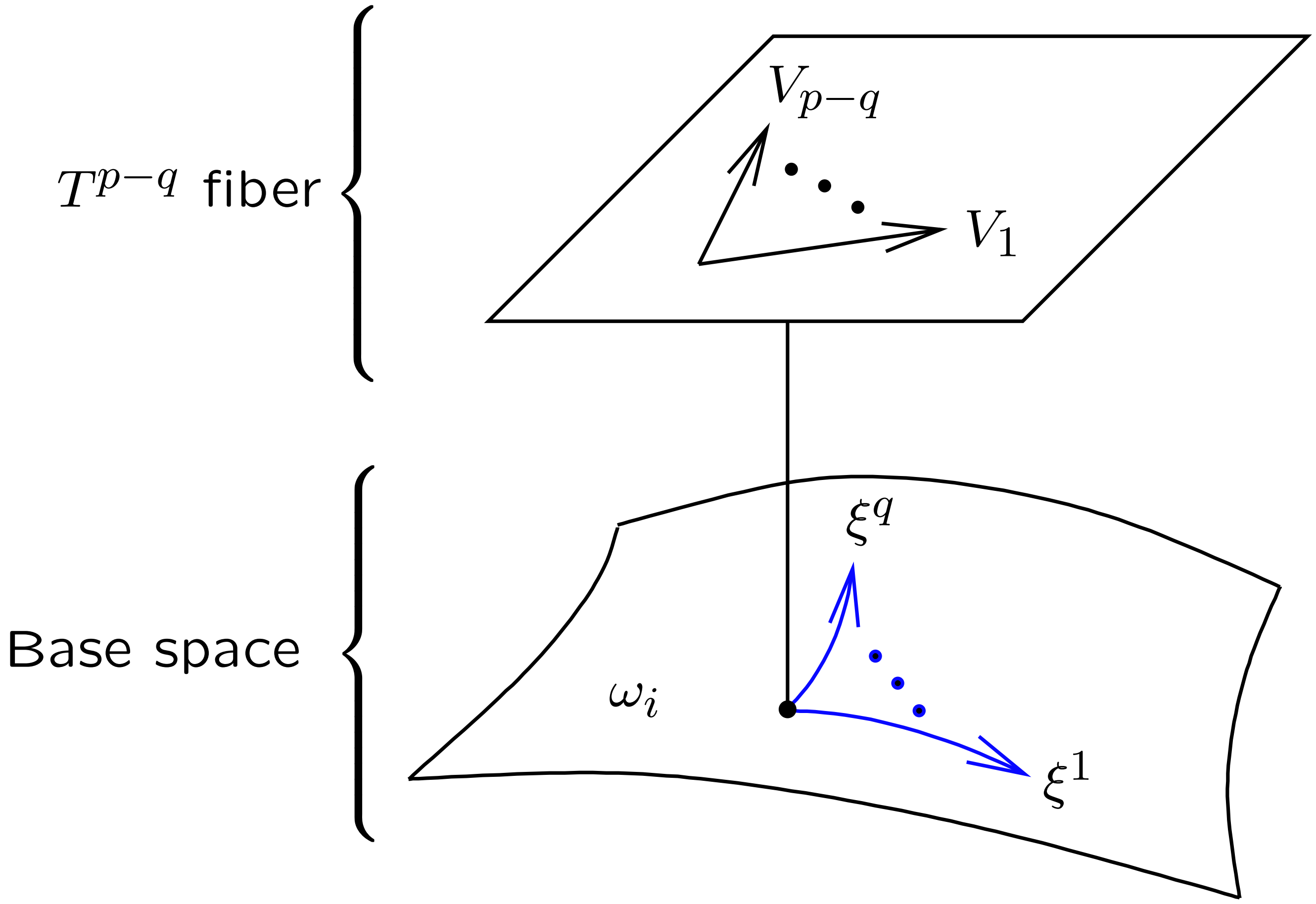
This will impose restrictions on which generalized 3-cycles will be allowed.

We will call these restrictions the **fiber condition**.

To ensure that **whenever we have an integral over a dual direction**, the associated **coordinate parametrizes a circle**, we insist that we can write Σ as a **torus** fibered over some **base space**.

We then insist that the non-zero vectors parts of the V_i are a **basis for the tangent bundle of the torus fiber**, while the **forms** only **live on the base space**.

Furthermore, we impose that nothing depends on the T^{p-q} fiber.



$$\int f(\xi^\mu)\, d\xi^1 \wedge \ldots \wedge d\xi^q \wedge d\hat{\xi}_{q+1} \wedge \ldots \wedge d\hat{\xi}_p \ ,$$

Having defined a **generalized integral** and a **generalized 3-cycle**, we must now write down the **flux** that we wish to integrate over.

This should be a **scalar** under B-transformations, diffeomorphisms and T-duality.

The result, as given in the introduction, which will be motivated by the examples, is given by

$$\mathsf{H}(\mathsf{V}_1, \mathsf{V}_2, \mathsf{V}_3) = -\,\mathsf{Nij}(\widetilde{\mathsf{V}}_1, \widetilde{\mathsf{V}}_2, \widetilde{\mathsf{V}}_3) \ ,$$

where

$$\widetilde{\mathsf{V}} = \mathsf{GV} \ .$$

The complete formula for the flux is

$$\int_\Sigma \mathsf{H} \equiv \int \mathsf{H}(\mathsf{V}_1, \mathsf{V}_2, \mathsf{V}_3)\ \mathsf{V}_1 \wedge \mathsf{V}_2 \wedge \mathsf{V}_3 \ .$$

Example 1: Ordinary H-flux

The simplest case to examine is when the vector part of the vielbeins vanishes:

$$\mathsf{V}_i = \begin{pmatrix} 0 \\ \omega_i \end{pmatrix} .$$

As we mentioned before, $\mathsf{G}(\mathsf{V}_i, \mathsf{V}_j) = \delta_{ij}$ implies that

$$\omega_{i\mu}\omega_{j\nu}\, g^{\mu\nu} = \delta_{ij},$$

so that the ω_i are **vielbeins in the ordinary sense**. Plugging the V_i into H gives

$$\mathsf{H}(\mathsf{V}_1, \mathsf{V}_2, \mathsf{V}_3) = \omega_1^\mu \omega_2^\nu \omega_3^\rho H_{\mu\nu\rho} \ .$$

Hence, the flux integral becomes

$$\int_\Sigma \mathsf{H}(\mathsf{V}_1, \mathsf{V}_2, \mathsf{V}_3)\, \mathsf{V}_1 \wedge \mathsf{V}_2 \wedge \mathsf{V}_3 = \int_\Sigma \left(\omega_1^\mu \omega_2^\nu \omega_3^\rho H_{\mu\nu\rho}\right) \omega_{1\gamma}\omega_{2\beta}\omega_{3\tau}\, d\xi^\gamma \wedge d\xi^\beta \wedge d\xi^\tau \ .$$

Noting again that the ω_i are just ordinary vielbeins, this reduces to

$$\int_\Sigma H \ ,$$

which is the standard formula for 3-form flux. Note that we did not need to worry about the **fiber condition** since $V_i = 0$.

Example 2: Geometric fluxes

Geometric fluxes arise from **T-dualizing** spaces with H-flux [Bouwknegt, Evslin, Mathai]. We suppose that Σ has one **killing vector** V that generates a circle bundle. We then pick

$$\mathsf{V}_{1,2} = \begin{pmatrix} 0 \\ \omega_{1,2} \end{pmatrix} \ , \qquad \mathsf{V}_3 = \begin{pmatrix} V \\ i_V B \end{pmatrix} \ .$$

One might have suspected that the correct choice for V_3 would be a pure vector, however this would not transform correctly under B-transformations and typically would not be a section of E_Σ globally.

The property $\mathsf{G}(\mathsf{V}_i, \mathsf{V}_j) = \delta_{ij}$ becomes

$$V^\mu V^\nu g_{\mu\nu} = 1 \ , \qquad \omega_{i\mu}\omega_{j\nu} g^{\mu\nu} = \delta_{ij} \ .$$

It is convenient to pick one of our coordinates, ξ^3 to be the circle coordinate with period 1, so that

$$V = \frac{1}{\sqrt{g_{33}}} \frac{\partial}{\partial \xi^3} \ .$$

A straightforward computation yields

$$\mathsf{H}(\mathsf{V}_1, \mathsf{V}_2, \mathsf{V}_3) = \omega_1^\mu \omega_2^\nu (V_{\nu,\mu} - V_{\mu,\nu}) \ .$$

The measure factor $\mathsf{V}_1 \wedge \mathsf{V}_2 \wedge \mathsf{V}_3$ gives

$$\frac{1}{\sqrt{g_{33}}} \omega_{1\mu} \omega_{2\nu} d\xi^\mu \wedge d\xi^\nu \wedge d\hat{\xi}_3 \ .$$

Putting everything together, our flux takes the form,

$$\int \frac{1}{\sqrt{g_{33}}} \omega_1^\mu \omega_2^\nu (V_{\nu,\mu} - V_{\mu,\nu})\, \omega_{1\mu} \omega_{2\nu} d\xi^\mu \wedge d\xi^\nu \wedge d\hat{\xi}_3 \ .$$

This can be rewritten as

$$\int \omega_1^\mu \omega_2^\nu (dA)_{\mu\nu}\, \omega_{1\mu} \omega_{2\nu} d\xi^\mu \wedge d\xi^\nu \wedge d\hat{\xi}_3 \ ,$$

where

$$A_\mu = (\sqrt{g_{33}})^{-1} V_\mu = \frac{g_{\mu 3}}{g_{33}} \ .$$

Note that A_μ is just the **connection** on the **circle bundle** generated by the vector field V.

Since we have picked the length of our circle-coordinate to be one, we may simply drop the $d\hat{\xi}_3$. The integral then reduces to

$$\int \omega_1^\mu \omega_2^\nu (dA)_{\mu\nu}\, \omega_{1\mu}\omega_{2\nu}\, d\xi^\mu \wedge d\xi^\nu = \int dA \ ,$$

where the integral is performed over the **base** of the **circle-fibration**. This gives the **first Chern-class** of the circle bundle, which is the **geometric flux**.

Example 3: Non-geometric Q-flux

One kind of non-geometric flux, known as Q-flux, which has been studied recently is associated with a so-called β-transformation. [Mellerman, McGreevy, Williams; Kachru, Schultz, Tripathy; Hull; Lowe, Nastase, Ramgoolam; Mathai, Rosenberg; Shelton, Taylor, Wecht; IE, Hashimoto; Benmachichi, Grimm]

A β-transformation is the double T-dual of a B-transformation. It acts on generalized vectors as [Gualtieri]

$$\mathsf{V} \to e^{\beta}\mathsf{V} \equiv \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} V \\ \omega \end{pmatrix},$$

where β is an antisymmetric matrix.

A **Q-space** is a T^2 fibered over an S^1 in which, when one goes around the S^1, one performs a β-transformation.

In order to find global sections of E_Σ, we should look for vielbeins that are not affected by β-transformations.

These are given by generalized vectors whose **1-form part vanishes**.

For definiteness, let our space be a T^2 with coordinates $\xi^{2,3}$ fibered over an S^1 with coordinate ξ^1. Consider a metric and B-field of the form,

$$g = \begin{pmatrix} 1 & \\ & g_{ab} \end{pmatrix} \ , \qquad B = \begin{pmatrix} 0 & \\ & B_{ab} \end{pmatrix} \ ,$$

where a, b run over 2, 3 and nothing depends on the coordinates $\xi^{2,3}$ of the T^2. We take for our vielbeins,

$$\mathsf{V}_1 = d\xi^1 \ , \qquad \mathsf{V}_2 = v_1^a \frac{\partial}{\partial \xi^a} \ , \qquad \mathsf{V}_3 = v_3^a \frac{\partial}{\partial \xi^a} \ .$$

The property $\mathsf{G}(\mathsf{V}_i, \mathsf{V}_j) = \delta_{ij}$ is now quite complicated:

$$v_i^a (g - B g^{-1} B)_{ab} v_j^b = \delta_{ij} \ .$$

Nonetheless, it is straightforward to find an appropriate pair of v's and substitute it into H, yielding

$$\mathsf{H}\ \mathsf{V}_1 \wedge \mathsf{V}_2 \wedge \mathsf{V}_3 = \left[\frac{\partial}{\partial \xi^1} \mathsf{Re} \left(\frac{1}{\tau} \right) \right] d\xi^1 \wedge d\hat{\xi}_2 \wedge d\hat{\xi}_3 \ ,$$

where we have defined $\tau = B_{12} + i\sqrt{g}$. Assuming that the $\xi^{2,3}$ coordinates run from 0 to 1, we can perform the integral over them trivially, yielding...

The flux,

$$Q\text{-flux} = \int \mathsf{H}\ \mathsf{V}_1 \wedge \mathsf{V}_2 \wedge \mathsf{V}_3 = \int d\xi^1 \frac{\partial}{\partial \xi^1} \mathsf{Re}\left(\frac{1}{\tau}\right) .$$

To illuminate the meaning of this expression, we note that a β-transformation acts as

$$\tau \to \frac{\tau}{1+\beta\tau} , \qquad (\spadesuit)$$

which takes

$$\mathsf{Re}\left(\tau^{-1}\right) \to \mathsf{Re}\left(\tau^{-1}\right) + \beta \ .$$

Since the flux is an integral of a total derivative, the Q-flux is given by the β-transformation that maps the top to the bottom.

Since $(\spadesuit)$ must be an element of $SL(2,\mathbb{Z})$, this gives an integer.

The generalized connection and the flux

We discuss a **generalized connection** that acts on generalized vectors and its relation to the generalized flux.

Although the flux H often arises as a **torsion of a connection**, computing the analogue of the torsion of the generalized connection, we see that it vanishes.

However, we find that the flux H arises from an object very similar to the torsion.

For clarity, it is useful to introduce an **index notation**.

We denote a generalized vector by V^I where the index I runs over the tangent indices followed by the cotangent indices.

The indices can be raised and lowered using the metric,

$$\mathsf{X}_{IJ} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \mathsf{X}^{IJ} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Note that a lowered index, as in V_I simply runs over the cotangent indices first followed by the tangent indices.

The matrix G has index structure $\mathsf{G}^I{}_J$. The lowered matrix G_{IJ} is the positive definite metric $\mathsf{G}(\,,\,)$.

The raised matrix G^{IJ} is, as one would like, the inverse of G_{IJ} so that $\mathsf{G}_{IJ}\mathsf{G}^{JK} = \delta_I^K$.

This follows from the basic property that $\mathsf{G}^2 = 1$.

The goal of this section is to write down a generalized connection D^I which, when acting on vectors,

$$\mathsf{D}^I \mathsf{V}^J \ ,$$

gives a two index object covariant under diffeomorphisms, B-transformations and T-duality.

Note that this is **not a connection** in the **ordinary sense**, since it allows one to **take derivatives with respect to the T-dual coordinates**.

To define the **generalized connection**, we begin by defining an **ordinary connection** on E. This connection will be invariant under diffeomorphisms and B-transformations, but will not be invariant under T-duality.

We take the connection to be of the form

$$D_\mu = \partial_\mu + \Omega_\mu \ ,$$

where Ω_μ is a matrix ${\Omega_\mu}^I{}_J$ which acts on the generalized vector indices.

When the B-field vanishes, it is very natural to take the connection to have the form

$$D_\mu\Big|_{B=0} = \begin{pmatrix} \nabla_\mu & 0 \\ 0 & \hat{\nabla}_\mu \end{pmatrix},$$

where ∇_μ is the Levi-Civita connection on vectors and $\hat{\nabla}_\mu$ is the Levi-Civita on 1-forms.

When $B \neq 0$, one can partially fix the form of D_μ by demanding that it annihilate both X^{IJ} and G^{IJ} and that it transform covariantly under B-transformations.

This unfortunately is not enough to completely determine the connection, as one is still left with a one-parameter family of possible connections:

$$\begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \left[\begin{pmatrix} \nabla_\mu & 0 \\ 0 & \hat{\nabla}_\mu \end{pmatrix} + a \begin{pmatrix} 0 & \frac{1}{2} g^{-1} H_\mu g^{-1} \\ \frac{1}{2} H_\mu & 0 \end{pmatrix} \right] \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix}.$$

Here we have used the shorthand H_μ for $H_{\mu\nu\rho}$ where ν and ρ are treated as matrix indices.

To fix an appropriate choice for a, it is useful to turn to string theory for guidance.

Recall that in the fermionic terms of the $N = 1$ string action the kinetic terms use the connection

$$\nabla^{\pm}_{\mu} = \nabla_{\mu} \pm \tfrac{1}{2} g^{-1} H_{\mu} \ ,$$

where we take $+$ for the right moving fermions and $-$ for the left moving fermions.

This connection is known as the **Bismut connection** and is relevant for a number of applications in generalized complex geometry [Hitchin; Gualtieri]

We can now fix the form of D_{μ} by insisting that if $\mathsf{V} \in \mathsf{C}^{\pm}$ that

$$\pi(D_{\mu}\mathsf{V}) = \nabla^{\pm}_{\mu}\pi(\mathsf{V}) \ .$$

In other words, the **covariant derivative just acts as the Bismut connection on the vector part of** V.

This extra condition fixes $a = 1$ and gives the **lift** of the **Bismut connection** to **generalized geometry**:

$$D_\mu \equiv \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \begin{pmatrix} \nabla_\mu & \frac{1}{2} g^{-1} H_\mu g^{-1} \\ \frac{1}{2} H_\mu & \hat{\nabla}_\mu \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix} .$$

This connection has nice properties under T-duality. Suppose the x-direction parametrizes a circle and that neither g nor B depends on x. Then we find

$$D^\mu \xrightarrow{\mathsf{T}_x} \mathsf{T}_x D^\mu \mathsf{T}_x \ , \qquad \mu \neq x \ ,$$
$$D^x \xrightarrow{\mathsf{T}_x} \mathsf{T}_x \left(B_{x\sigma} D^\sigma - \mathsf{G} g_{x\sigma} D^\sigma \right) \mathsf{T}_x \ .$$

The matrices T_x are the elements of $SO(d,d)$ which represent T-duality along the x direction.

Since T-duality switches vectors with forms, it is natural to take for the generalized connection,

$$\mathsf{D}^I = \begin{pmatrix} D^\mu \\ -\mathsf{G} D_\mu + B_{\mu\sigma} D^\sigma \end{pmatrix} ,$$

Parallel transport and torsion

We define the **parallel transport** of V_2 along V_1 by

$$\mathsf{D}_{\mathsf{V}_1}\mathsf{V}_2^K = \mathsf{X}_{IJ}\mathsf{V}_1^I\,\mathsf{D}^J\mathsf{V}_2^K \ .$$

From this, one finds the rather strange "Leibniz rule",

$$\mathsf{D}_{\mathsf{V}_1}(f\mathsf{V}_2) = f\mathsf{D}_{\mathsf{V}_1}\mathsf{V}_2 + [\pi(\widetilde{\mathsf{V}}_1)(f)]\mathsf{V}_2 - [\pi(\mathsf{V}_1)(f)]\widetilde{\mathsf{V}}_2 \ .$$

The standard definition of the **torsion** is

$$\mathsf{T}(\mathsf{V}_1,\mathsf{V}_2) = \mathsf{D}_{\mathsf{V}_1}\mathsf{V}_2 - \mathsf{D}_{\mathsf{V}_2}\mathsf{V}_1 - [\mathsf{V}_1,\mathsf{V}_2] \ .$$

A nice choice for the bracket, $[,]$, which makes T into a tensor, is given by

$$[\mathsf{V}_1,\mathsf{V}_2] = \mathsf{G}[\widetilde{\mathsf{V}}_1,\widetilde{\mathsf{V}}_2]_C - \mathsf{G}[\mathsf{V}_1,\mathsf{V}_2]_C \ .$$

A straightforward, but tedious computation of $\mathsf{T}(\mathsf{V}_1,\mathsf{V}_2)$ reveals that

$$\mathsf{T}(\mathsf{V}_1,\mathsf{V}_2) = 0 \ ,$$

so that, in this sense, **the torsion vanishes!**

Consider, however, the **“torsion-like quantity”**,

$$-\frac{1}{3}\left[\langle(\mathsf{D}_{\mathsf{V}_1}\mathsf{V}_2),\mathsf{V}_3\rangle-\langle(\mathsf{D}_{\mathsf{V}_2}\mathsf{V}_1),\mathsf{V}_3\rangle+\mathsf{cyclic}\right]\ .$$

Using $\mathsf{T}(\mathsf{V}_1,\mathsf{V}_2)=0$, this becomes

$$\mathsf{H}(\mathsf{V}_1,\mathsf{V}_2,\mathsf{V}_3)-\frac{1}{3}[\mathsf{G}([\mathsf{V}_1,\mathsf{V}_2]_C,V_3)+\mathsf{cyclic}]\ .$$

Notice that for V’s which are appropriate for a generalized 3-cycle, we would have $[\mathsf{V}_i,\mathsf{V}_j]_C=0$, so that the second term would vanish.

This is how we originally found the flux formula.

How is this anything like a torsion? Consider, for the Bismut connection, for example, we would find

$$\nabla^{\pm}_{V_1}V_2^{\mu}V_{3\mu}-\nabla^{\pm}_{V_2}V_1^{\mu}V_{3\mu}=\pm H(V_1,V_2,V_3)+[V_1,V_2]^{\mu}V_{3\mu}$$

yielding the torsion plus a term that vanishes provided $[V_1,V_2]=0$.

Comments on the generalized connection

Before concluding, we give a few observations about the action of generalized connection on tensors.

For A a generalized vector, we have the following identity,

$$\mathsf{G}^I{}_J \mathsf{D}^J \mathsf{A}^K + \mathsf{D}^I(\mathsf{G}^K{}_L \mathsf{A}^L) = 0 \ .$$

This implies that the **index** on the **generalized connection** lives in the **opposite half of the splitting** as the **index** of the **vector** it is differentiating.

Because of the G in the definition of D^I,

$$\mathsf{D}^I = \begin{pmatrix} D^\mu \\ -\mathsf{G}D_\mu + B_{\mu\sigma}D^\sigma \end{pmatrix} ,$$

D^I does not satisfy the **Leibniz rule** when acting on products of vectors unless all of the vectors live in C^+ or all live in C^-.

This implies that it is not meaningful to speak of differentiating a tensor $\mathsf{T}^{I_1 I_2 \ldots I_n}$ unless it satisfies

$$\forall i, j \qquad \mathsf{G}^{I_i}{}_J \mathsf{T}^{I_1 \ldots I_{i-1} J I_{i+1} \ldots I_n} = \mathsf{G}^{I_j}{}_J \mathsf{T}^{I_1 \ldots I_{j-1} J I_{j+1} \ldots I_n} \ .$$

Because differentiation gives an index in the opposite half of the splitting, this property is not preserved under action by D^I.

This makes it very difficult to construct a curvature of the generalized connection.

Summary

We've shown that in **generalized geometry**, we should replace

$$\int_\Sigma H \longrightarrow \int_\Sigma \mathsf{H} \ .$$

We've also shown that H arises very naturally as a **"torsion"** of the generalized connection,

$$\mathsf{D}^I = \begin{pmatrix} D^\mu \\ -\mathsf{G}D_\mu + B_{\mu\sigma}D^\sigma \end{pmatrix} \ .$$

Together this suggests very strongly that

$$\mathsf{H}(\mathsf{V}_1, \mathsf{V}_2, \mathsf{V}_3) = -\,\mathsf{Nij}(\widetilde{\mathsf{V}}_1, \widetilde{\mathsf{V}}_2, \widetilde{\mathsf{V}}_3)$$

is the generalized analogue of the three form H.

Open problems

1. In our construction of the **generalized flux integral**, we relied heavily on what we called **fiber condition**.

   This condition was required to ensure that we could give a sensible definition of integration over a generalized 3-cycle.

   It seems likely, however, that the spaces on which integration is well-defined could be extended.

   Currently, for example, our definition is not broad enough to handle spaces where coordinates and dual coordinates mix on the torus fibers and we are, thus, not able to realize a full $SO(d,d;\mathbb{Z})$ symmetry for our definition of integration.

2. Although in the examples we were able to show that the integral of H over a generalized 3-cycle was always a **topological quantity** and in fact an **integer**, it would be nice to have proof of this **in the framework of generalized geometry**.

3. Our discussion of the generalized connection seems far from complete. There is already a well-established connection which acts on **pure spinors**, and it would be interesting to try to connect the two. It also seems quite interesting to try to understand whether there is a natural notion of the **curvature of the connection**.

4. It would be nice to give a **stringy** derivation of the **flux formula**. The string action can already be written in a generalized form,

$$S = \frac{1}{4}\int \mathsf{G}_{IJ}\mathsf{Z}^I \wedge *\mathsf{Z}^J + \mathsf{B}_{IJ}\mathsf{Z}^I \wedge \mathsf{Z}^J \ ,$$

where B_{IJ} is the canonical anti-bracket of E given by the matrix,

$$\mathsf{B}_{IJ} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and $\mathsf{Z} = dX^\mu + \Omega_\mu$ where Ω_μ is an auxiliary one-form on the worldsheet as well as in spacetime. It would be quite nice if we could replace the B term with a **WZW-term** involving H, but we have not yet found a way to do so.

The End