Generalized $N = 1$ orientifold compactifications

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based on: [hep-th/0602241] Iman Benmachiche, TWG
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Introduction and Motivation

- Phenomenology

String Theory $\longrightarrow$ Four-dimensional $N = 1$ Supergravity

- Address moduli problem: generate potential for massless scalar fields due to background fluxes and non-Calabi-Yau geometries

- Explore generic features of $N = 1$ compactifications – Landscape of String vacua?

- Complete dualities in the presence of background flux

- Four-dimensional gauge theory and specific models
A realization in Type II String theory:

- minimal supersymmetry: background $M_{1,3} \times M_6$
  $M_6$ – special manifold

- moduli stabilization:
  background fluxes
  non-Calabi-Yau geometry

- non-Abelian gauge groups:
  space-time filling D-branes

⇒ consistency: orientifold planes
Outline of the Talk

• Conditions on supersymmetric theories and vacua

• Some basics on generalized complex geometry

• Orientifold projection and $N = 1$ spectrum

• $N = 1$ chiral field space: Kähler potential and superpotential

• Mirror symmetry / T-duality with fluxes: a conjecture
Conditions on supersymmetric theories and vacua
Condition for supersymmetric \( D = 4 \) theory: (Type IIB example)

- metric ansatz: 
  \[
  ds_{10}^2 = e^{A(y)} \eta_{\mu \nu} dx^\mu dx^\nu + g_{ij}(y) dy^i dy^j
  \]

- Two ten-dimensional gravitinos \( \Psi^1, \Psi^2 \) decompose as:
  \[
  \Psi^i_\mu = \psi^i_{+ \mu} \otimes \eta^i_+ + \psi^i_{- \mu} \otimes \eta^i_- \\
  \psi^1_\mu, \psi^2_\mu \text{ become four-dimensional gravitinos}
  \]
  \[
  \Psi^i_m = \text{four-dimensional fermionic modes for other multiplets}
  \]

- Effective four-dimensional theory possesses \( N = 2 \) susy

  \Rightarrow \text{two globally defined spinors } \eta^1 \text{ and } \eta^2 \text{ on } M_6 \text{ exist,}

  \text{they locally or globally coincide}

- \( \eta^1, \eta^2 \) reduce structure group \( SO(6) \) of \( T \! M_6 \):
  \[
  4 \rightarrow 1 + 3
  \]
  \[
  \eta^1: \quad SO(6) \leftrightarrow SU(3)_1 \\
  \eta^2: \quad SO(6) \leftrightarrow SU(3)_2
  \]

Graña, Louis, Waldram
Supersymmetry conditions on $D = 10$ flux-background

⇒ differential conditions relating $\eta^1$ and $\eta^2$ to the NS-NS and R-R background flux

Behrndt, Cvetic, Liu; Lüst, Tsimpis; Witt; Graña, Minasian, Petrini, Tomasiello

• vanishing of the gravitino and dilatino variations:

$$\delta \psi^i_M = \nabla_M \epsilon^i + (\text{Flux})_M \epsilon^i = 0$$

$$\delta \lambda^i = (\partial \phi) \epsilon^i + (\text{Flux}) \epsilon^i = 0$$

• Susy spinors $\epsilon^i$ decompose as:

$$\epsilon^i = \zeta^i_+ \otimes \eta^i_+ + \zeta^i_- \otimes \eta^i_-$$

(Type IIB example)

• Susy conditions imply:

$$\nabla_m \eta^1 = (\text{Flux})^1_m, \quad \nabla_m \eta^2 = (\text{Flux})^2_m$$

Examples for $\mathcal{M}_6$:

– Calabi-Yau manifolds: $\eta = \eta^1 = \eta^2$ and $\nabla_m \eta = 0$

– Manifolds with $SU(3)$ structure: $\eta = \eta^1 = \eta^2$, but $\nabla_m \eta \neq 0$

– Manifolds with $\eta^1 = \eta^2$ only locally: manifolds with $SU(3) \times SU(3)$ structure

new framework: Generalized complex geometry

Hitchin, Gualtieri, Witt
Some basics on generalized complex geometry
Central object: \( T \oplus T^* \equiv T\mathcal{M}_6 \oplus T^*\mathcal{M}_6 \) generalized tangent bundle

- \( T \oplus T^* \) takes the role of tangent bundle \( T \) in standard geometry:

  recall: metric \( g_{ij} \) on \( \mathcal{M}_6 \) is a map \( T \times T \to \mathbb{R} \)

  almost complex structure \( I^j_i \) on \( \mathcal{M}_6 \) defines a split \( T_C = T^{(1,0)} \oplus T^{(0,1)} \)

  \[ G_{IJ} = \begin{pmatrix} -g^{-1}B & g^{-1} \\ g - Bg^{-1}B &Bg^{-1} \end{pmatrix} \]

  \( \Rightarrow \) generalized almost complex structure \( J^I_I \)

  defines split \( (T \oplus T^*)_C = L^{(1,0)} \oplus L^{(0,1)} \)

  \( \mathcal{J}L^{(1,0)} = iL^{(1,0)} \quad \mathcal{J}L^{(0,1)} = -iL^{(0,1)} \quad \mathcal{J}^2 = -1 \)
$T \oplus T^*$ has more structure:

- **natural inner product on** $T \oplus T^*$: $V, W \in T \quad \eta, \zeta \in T^*$

  $$(V + \eta, W + \zeta) = (V \eta) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} W \\ \zeta \end{pmatrix} = \eta(W) + \zeta(V)$$

  \Rightarrow **generalized structure group** of $T \oplus T^*$ is $SO(6, 6)$

- $SO(6, 6)$ induces a Clifford action on differential forms $\Phi$:

  $$(V + \eta) \cdot \Phi = V \downarrow \Phi + \eta \wedge \Phi$$

  Indeed we have $\{V + \eta, W + \zeta\} \cdot \Phi = (V + \eta, W + \zeta)\Phi$.

**Differential forms $\Phi$ on $\mathcal{M}_6$ are spinors of $SO(6, 6)$:**

$$S^{\text{ev}} = \Lambda^{\text{ev}} T^* \otimes \sqrt{\Lambda^6 T} \quad S^{\text{odd}} = \Lambda^{\text{odd}} T^* \otimes \sqrt{\Lambda^6 T}$$
$SO(6)$ spinors $\eta^1$, $\eta^2$ vs. $SO(6,6)$ spinors $\Phi$

- Globally defined even and odd forms naturally associated to $\eta^1$ and $\eta^2$:

$$\Phi^{ev} = \sum_{i=1}^{6} \eta_{+}^2 \gamma_{m_1...m_i} \eta_{+}^1 \, dx^{m_1} \wedge ... \wedge dx^{m_i}$$

$$\Phi^{odd} = \sum_{i=1}^{6} \eta_{-}^2 \gamma_{m_1...m_i} \eta_{+}^1 \, dx^{m_1} \wedge ... \wedge dx^{m_i}$$

- Conditions on $\eta^1$, $\eta^2$ translate into conditions on forms $\Phi^{ev/odd}$
• **pure spinors**: \[(V + \eta) \cdot \Phi^\text{ev/odd} = 0 \quad V + \eta \in L^\text{ev/odd} \subset T + T^*\]

\[L^\text{ev/odd}\] are of dimension six and isotropic

\[L^{(1,0)} = L \quad \Rightarrow \quad \Phi^\text{ev/odd}\] define two generalized almost complex structures

• **non-degeneracy**: \[
\langle \Phi^\text{ev}, \bar{\Phi}^\text{ev} \rangle \neq 0 \quad \langle \Phi^\text{odd}, \bar{\Phi}^\text{odd} \rangle \neq 0
\]

\[
\langle \cdot, \cdot \rangle \text{ are the Mukai pairings: e.g. for } \Phi^\text{ev} = \Phi_0 + \Phi_2 + \Phi_4 + \Phi_6
\]

\[
\langle \Phi^\text{ev}, \bar{\Phi}^\text{ev} \rangle = \Phi_0 \wedge \bar{\Phi}_6 - \Phi_2 \wedge \bar{\Phi}_4 + \Phi_4 \wedge \bar{\Phi}_2 - \Phi_6 \wedge \bar{\Phi}_0
\]

• **compatibility**: \[
\langle \Phi^\text{ev}, \bar{\Phi}^\text{ev} \rangle = \frac{3}{4} \langle \Phi^\text{odd}, \bar{\Phi}^\text{odd} \rangle \quad \text{and} \quad \langle \Phi^\text{odd}, (V + \eta) \cdot \Phi^\text{ev} \rangle = 0
\]

\[
\Phi^\text{ev/odd}\] reduce structure group of \(T \oplus T^*\) to \(SU(3) \times SU(3)\)

\[
\Rightarrow \quad M_6 \text{ is generalized almost complex manifold with } SU(3) \times SU(3) \text{ structure}
\]

flux background: \[
d\Phi^\text{ev} \neq 0 \quad d\Phi^\text{odd} \neq 0\]
Important example: $SU(3)$ structure manifolds

$$
\Phi^{ev} = e^{iJ}, \quad \Phi^{odd} = \Omega
$$

where:

- $J$ is a globally defined $(1, 1)$ form
- $\Omega$ is a globally defined $(3, 0)$ form

$$
\Rightarrow \quad J, \Omega \text{ define the } SU(3) \text{ structure: } J \wedge J \wedge J = \frac{3i}{4} \Omega \wedge \bar{\Omega}
$$

$$
J \wedge \Omega = 0
$$

Remark on NS-NS $B$-field: note that world-sheet coupling is $B + iJ$

**include by a $B$-transform:**

| $\Phi$ | $\Phi_B \equiv e^B \wedge \Phi$ |

- $B$-transform is a symmetry of Mukai pairings: $\langle \Phi_B, \Psi_B \rangle = \langle \Phi, \Psi \rangle$

- $B$-transform is $SO(6, 6)$ rotation on $T \oplus T^*$: maps a pure spinor $\Phi$ into new pure spinor $\Phi_B$

- $B$ has a non-trivial background flux $H = dB \quad \Rightarrow \quad \Phi_B$ obtained by ‘twisting with a gerbe’
Orientifold projection and $N = 1$ spectrum
The orientifold projection (Type IIA example):

Bosonic Type IIA spectrum

| NS-NS: | $\phi, g_{MN}, B_2$ | R-R: | $C^{\text{odd}} \equiv C_1 + C_3 + C_5 + C_7 + C_9$ |

- mod out (gauge-fix) discrete symmetries of the string theory:

  1) world sheet parity $\Omega_p$
  \[{\mathcal{O} = (-1)^F L \Omega_p \sigma^*}

  2) geometric symmetry $\sigma$ of $M_6$: $\sigma^2 = 1$ (identity on $M_{3,1}$)

- demand $N = 1$ supersymmetry

  \[
  \lambda(\omega_{2n}) = (-1)^n \omega_{2n} \quad \lambda(\omega_{2n-1}) = (-1)^n \omega_{2n-1}
  \]

  \[
  \sigma^* \Phi^{\text{odd}} = \lambda(\overline{\Phi}^{\text{odd}}) \quad \sigma^* \Phi^{\text{ev}} = \lambda(\Phi^{\text{ev}})
  \]

  Benmachiche, TWG

  Calabi-Yau case: $\sigma$ is anti-holomorphic and isometric involution – $O6$ planes.

- truncate spectrum such that: $\mathcal{O}(\text{Field}) = \text{Field}$

  \[
  \sigma^* \phi = \phi \quad \sigma^* B_2 = -B_2 \quad \sigma^* C^{\text{odd}} = \lambda(C^{\text{odd}})
  \]
Four-dimensional spectrum:

- \( \mathcal{M}_6 \) generalized complex manifold: two-grading of forms
  \[
  \Lambda^{ev} T^* \cong S^{ev} \quad \quad \Lambda^{odd} T^* \cong S^{odd}
  \]

- \( \mathcal{M}_6 \) possesses orientifold symmetry \( \lambda \sigma^* \): four-grading of forms
  \[
  \Lambda^{ev} T^* = \Lambda^{ev}_+ \oplus \Lambda^{ev}_- \quad \quad \Lambda^{odd} T^* = \Lambda^{odd}_+ \oplus \Lambda^{odd}_-
  \]

Orientifold: restrict ten-dimensional fields to appropriate eigenspaces of \( \sigma^* \)

- **R-R sector**: four eigenspaces correspond to four fields in \( D = 4 \):
  
<table>
<thead>
<tr>
<th>Eigenspace</th>
<th>Fields</th>
</tr>
</thead>
<tbody>
<tr>
<td>scalars</td>
<td>( C^{odd}_{(0)} = C^{odd}</td>
</tr>
<tr>
<td>vectors</td>
<td>( C^{odd}_{(1)} = C^{odd}</td>
</tr>
<tr>
<td>two-forms</td>
<td>( C^{odd}_{(2)} = C^{odd}</td>
</tr>
<tr>
<td>three-forms</td>
<td>( C^{odd}_{(3)} = C^{odd}</td>
</tr>
</tbody>
</table>

So far: Infinite set of scalars, two-forms as well as vectors, three-forms related by duality condition on the ten-dimensional field strengths:

\[
* F^{ev} = \lambda(F^{ev}) \quad \quad F^{ev} = dC^{odd} + H \wedge C^{odd}
\]
• **NS-NS sector**

  We define

  \[ \varphi^{\text{odd}} = e^{-\phi} e^{B_2} \wedge \Phi^{\text{odd}} \quad \varphi^{\text{ev}} = e^{B_2} \wedge \Phi^{\text{ev}} \]

  Four-dimensional graviton and \( \varphi^{\text{ev/odd}} \) encode all degrees of freedom in the NS-NS sector.

  Not all degrees of freedom in \( \varphi^{\text{ev/odd}} \) are independent:

  - \( \varphi^{\text{odd}} \) pure spinor and non-degeneracy \( \Rightarrow \) \( \text{Im}(\varphi^{\text{odd}}) \) is function of \( \text{Re}(\varphi^{\text{odd}}) \) \( \text{Hitchin} \)

  - \( \varphi^{\text{ev}} \) complicated function \( \varphi^{\text{ev}}(t) \) of the true scalar deformations \( t \) of the generalized complex manifold (e.g. complex rescalings of \( \varphi^{\text{ev}} \) are unphysical) \( \text{Gualtieri; Graña,Louis,Waldram} \)

  (compare with Calabi-Yau case \( \varphi^{\text{ev}} = e^{B+iJ} = e^{t^a \omega_a} \))
Performing the Kaluza-Klein reduction

• What are the light modes on $SU(3)$ or $SU(3) \times SU(3)$ structure manifold?
  ⇒ determine a finite set of forms on $\mathcal{M}_6$ used in the Kaluza-Klein expansion:
  \[ \Delta_{\text{finite}} = \Delta^{\text{ev}} \oplus \Delta^{\text{odd}} \]
  $\Delta^{\text{finite}}$ often constructed to match mirrors of Calabi-Yau compactifications with fluxes

Gurrieri, Louis, Micu, Waldram; Tomasiello; Graña, Louis, Waldram; Benmachiche, TWG

• How to perform a Kaluza-Klein reduction with a warp factor?
  ⇒ restrict to regime of constant warping, large volume limit

Giddings, Kachru, Polchinski
$N = 1$ chiral field space: Kähler potential and superpotential
1. How do the the scalars combine into $N = 1$ chiral multiplets?

- correct D-brane couplings: combine

$$\varphi_c^{\text{odd}} \equiv e^{B_2} \wedge C^{\text{odd}} + i \text{Re}(\varphi^{\text{odd}}) \big|_{\Delta_+^{\text{odd}}}$$

linear in the $N = 1$ complex scaler fields

- $\varphi^{\text{ev}}(t)$ is holomorphic function of complex scalars $t^a$

2. What is the metric on the scalar field space?

- $N = 1$ susy $\rightarrow$ Kähler metric, i.e. $G_{AB} = \partial_A \bar{\partial}_B K$

- Kähler potential:

$$K(t, \varphi^{\text{odd}}_c) = -\ln \int_{\mathcal{M}_6} \langle \varphi^{\text{ev}}, \bar{\varphi}^{\text{ev}} \rangle - 2 \ln \int_{\mathcal{M}_6} \langle \varphi^{\text{odd}}, \bar{\varphi}^{\text{odd}} \rangle$$

- first term: as in $N = 2$ scale invariant functional on even forms

- second term: Kähler space inside the $N = 2$ quaternionic manifold, metric encoded by generalized Hitchin functional of $\text{Re}(\varphi^{\text{ev}})$
The $N = 1$ superpotential:

The background fluxes and non-Calabi-Yau geometry induce a potential for the scalar fields.

- We allow for non-trivial NS-NS background flux $H_3$ and R-R fluxes $F^{\text{ev}}$ on $\mathcal{M}_6$

\[
H_3 = \langle dB_2 \rangle_{\mathcal{M}_6}, \quad F^{\text{ev}} = \langle dC^{\text{odd}} \rangle_{\mathcal{M}_6}
\]

- The manifold with $SU(3) \times SU(3)$ structure generically has $d\varphi_c^{\text{odd}} \neq 0$ and $d\varphi^{\text{ev}} \neq 0$. This deviation from a Calabi-Yau manifold contributes to the potential.

- The induced superpotential can be derived by a fermionic reduction. \text{Benmachiche,TWG}

\[
W(t, \varphi_c^{\text{odd}}) = \int_{\mathcal{M}_6} \langle F^{\text{ev}} + d_H \varphi_c^{\text{odd}}, \varphi^{\text{ev}} \rangle
\]

Here $d_H = d + H_3 \wedge$ denotes the $H$-twisted differential. This superpotential reduces to the known cases on general Calabi-Yau and $SU(3)$ structure orientifolds. \text{TWG,Louis; Villadoro,Zwirner; Graña,Louis,Waldram}

- A similar analysis can be performed for type IIB set-ups: essentially exchanges the role of even and odd forms
Comments on the Hitchin functionals

- let $\rho$ be a real even or odd form on $\mathcal{M}_6$ \((\rho \in S^{\text{ev/odd}})\)
- define invariant quartic form of $\rho$

\[
q(\rho) = \text{Tr} \mu(\rho)^2 = \langle \Gamma_{MN} \rho, \rho \rangle \langle \Gamma^{MN} \rho, \rho \rangle
\]

- $\mu(\rho)(a) = \langle \sigma(a) \rho, \rho \rangle$ is moment map of $SO(6, 6)$, $\sigma(a)$ representation of $so(6, 6)$
- $\Gamma^M$ are Gamma matrices of $SO(6, 6)$

Proposition: $q(\rho) < 0 \iff \rho = \text{Re} \Phi$ for pure spinor $\Phi$ and $\langle \Phi, \bar{\Phi} \rangle \neq 0$

$\hat{\rho} = \text{Im} \Phi$ is unique up to ordering

$\Rightarrow$ on the open set of non-degenerate pure spinors define Hitchin functional

\[
H[\rho] = \int_{\mathcal{M}_6} \sqrt{-q(\rho)} = i \int_{\mathcal{M}_6} \langle \Phi, \bar{\Phi} \rangle
\]

- can express $\text{Im} \Phi(\text{Re} \Phi)$

\[
\partial_\rho H[\rho] = \hat{\rho}
\]
Some other applications of the Hitchin functionals

- **Variational problem:** Extrema of $H[\rho]$ for variations in a cohomology class correspond to generalized Calabi-Yau manifolds with $d\Phi = 0$ (Hitchin)

- $H[\rho]$ is entropy functional for an $N = 2$ black hole: on harmonic forms it is the Legendre transform of the pre-potential $F$ (Dijkgraaf, Gukov, Neitzke, Vafa)

- $H[\rho]$ might serve as a space-time action for topological string theory on $M_6$ (Dijkgraaf, Gukov, Neitzke, Vafa; Pestun, Witten)
Mirror symmetry / T-duality with fluxes: a conjecture
The Question:

What is the mirror dual of type IIB Calabi-Yau $O3/O7$ orientifolds with background fluxes?

- Mirror symmetry/T-duality is believed to map H-flux to a non-trivial mirror geometry.

- In type IIB Calabi-Yau orientifolds fluxes induce the Gukov-Vafa-Witten superpotential

$$W_{GVW} = \int_Y (F_3 - \tau H_3) \wedge \Omega(z)$$

- Let us restrict to a simple case: one complex structure modulus $z$

$F_3 = 0$ and $H_3 = m\alpha_1 + e\beta^1$

In large complex structure limit:

$$W_{GVW} = -\tau(ez + mz^2)$$

Has the H-flux $e$ and $m$ an $SU(3) \times SU(3)$ mirror geometry?
Perform mirror symmetry (T-dulality in three directions SYZ):

$H_3$ has maximally two legs into the T-dualized directions (the ‘Q-space’ Shelton, Taylor, Wecht)

- Mirror deformation due to electric flux $e$: The complex three-form $\Omega_3$ is not anymore closed. $SU(3)$ structure mirror (‘half-flat’)

  Kachru, Schulz, Tripathy, Trivedi; Gurrieri, Louis, Micu, Waldram; Fidanza, Minasian, Tomasiello

  $\Phi^{\text{odd}} = \Omega_3 
  \quad d\text{Re}(\Omega_3) \propto e$

- Mirror deformation due to magnetic flux $m$: A non-trivial one-form $\Omega_1$ on the mirror space is needed:

  $\Phi^{\text{odd}} = \Omega_1 + \Omega_3 + \Omega_5 
  \quad d\text{Re}(\Omega_1) \propto m$

Such a one-form is present on an appropriate $SU(3) \times SU(3)$ manifold!

Can use the superpotential calculated above

$$W_{SU(3) \times SU(3)} = \int \langle d\varphi_c^{\text{odd}}, \varphi^{\text{ev}}(t) \rangle = -N^0(e \, t + m \, t^2)$$

- Origin of $\Omega_1$? Recall $so(6, 6) \cong \Lambda^2 T^* \oplus \Lambda^2 T \oplus \text{End}T \Rightarrow \beta \in \Lambda^2 T$ two-vector

$\Omega_1 + \Omega_3 = e^{-\beta} \Omega_3 : \beta$ correspond to non-commutativity of $\mathcal{M}_6$ for open strings

Kapustin, Mathai, Rosenberg
Conclusions

• discussed compactification of type II supergravity on orientifolds of $SU(3) \times SU(3)$ manifolds
  – determined four-dimensional spectrum without finite truncation, NS-NS and R-R sector is encoded by specific odd and even forms on $\mathcal{M}_6$
  – Kähler potential consists of the two Hitchin functionals on $\mathcal{M}_6$
  – holomorphic superpotentials for the geometry due to fluxes and non-Calabi-Yau geometry

• commented on mirror symmetry/T-duality of flux compactifications
  – mirror spaces of Calabi-Yau compactifications with $H$-flux are generically generalized $SU(3) \times SU(3)$ manifolds

• open questions:
  – determination of light modes for non-Calabi-Yau compactifications
  – effective four-dimensional theories for warped compactifications
  – interesting compact examples with space-time filling D-branes
\( \text{N=2 moduli space} \)

Special Kähler (vector multiplets)

\( \mathcal{M}^{\text{SK}} \)

Quaternionic (hyper multiplets)

\( \mathcal{M} \)

\( \times \)

\( \text{N=1 moduli space} \)

\( \mathcal{M}^{\text{SK}} \) remains special Kähler (same complex structure)

\( \mathcal{M}^{\text{Q}} \) Kähler – half of Quaternionic