An overview of open string field theory

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Introduction

One of the strange features of string theory is that many of the things we take for granted in ordinary QFT are unknown.

In ordinary **QFT** we star with a **collection of fields** ϕ^i and an **action** $S(\phi^i)$

We then get all the **classical physics** of the system through

$$\delta S\Big|_{\phi^i_{\mathrm{cl}}} = 0$$

and **quantum physics** through evaluating correlators,

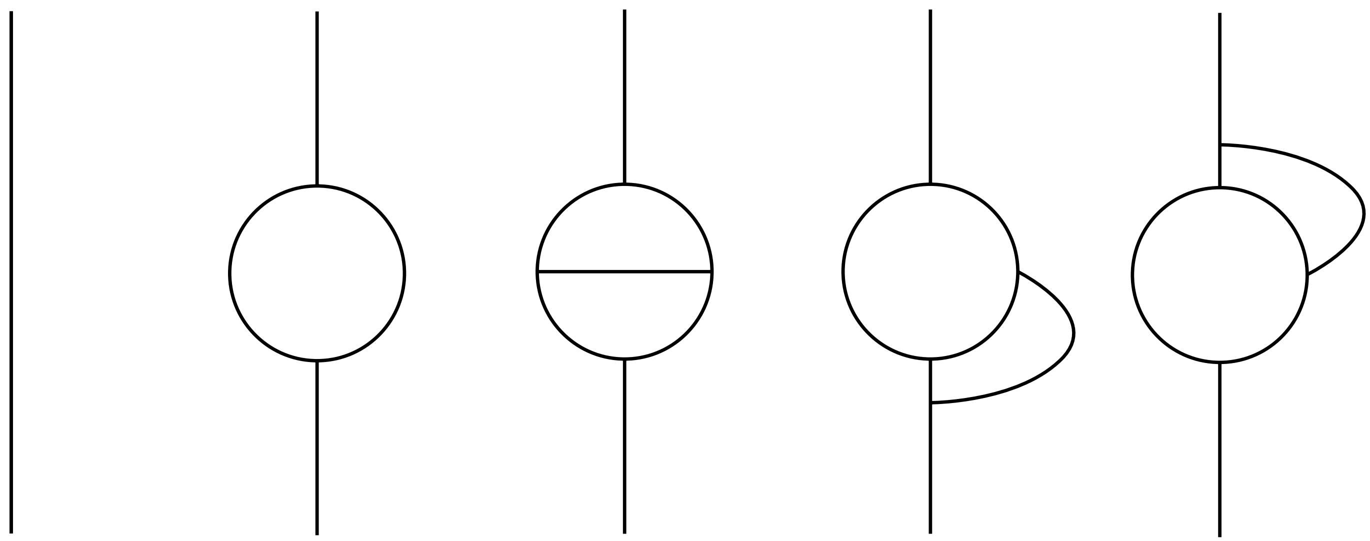
$$\langle f(\phi^i)\rangle = \int \mathcal{D}\phi^i f(\phi^i) \exp\left\{-S(\phi^i)\right\} \ .$$

Of course, evaluating these correlators exactly is typically quite hard, but we may write

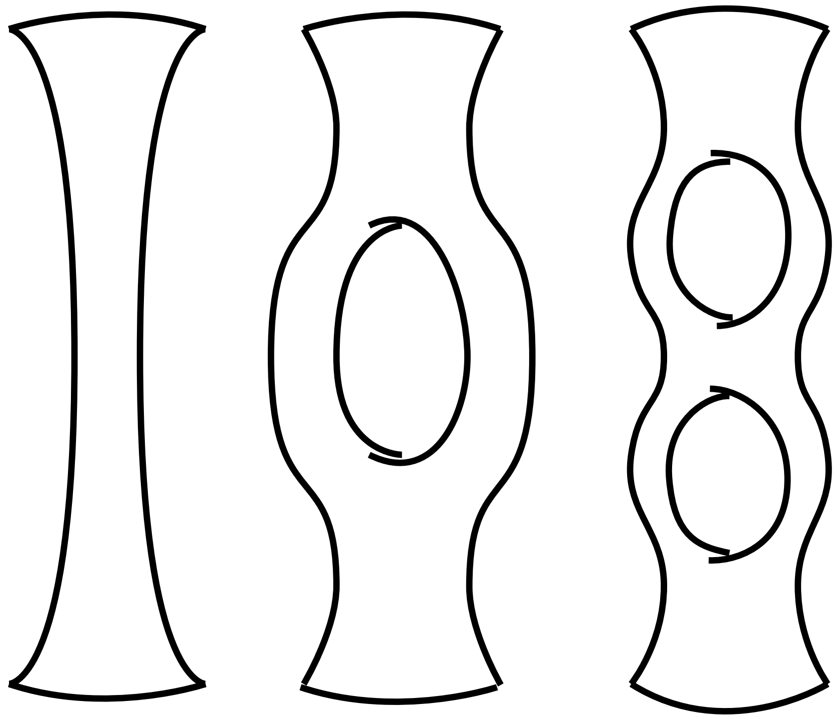
$$S = S_{\text{free}} + gS_{\text{int}} \ , \tag{1}$$

where g is **small**.

Expanding correlators in powers of g leads to an expansion,



In string theory, we do not start with the action, but instead with the diagrammatic expansion:



This allows us to compute scattering amplitudes of the various fields in the free string hilbert space

$$g_{\mu\nu}, \quad B_{\mu\nu}, \quad \phi, \quad \text{and lots of massive fields...}$$

We do not, a priori, have an action which generates this stringy expansion. We do, however, have a *very rough* correspondence

$$\text{Classical fields} \quad \Longleftrightarrow \quad \text{2-d field theories}$$

$$\text{Classical solution} \quad \Longleftrightarrow \quad \text{2-d conformal field theory}$$

This correspondence is quite useful for deriving low-energy effective actions.

One starts with a reference CFT (e.g. flat space with all other fields vanishing) and turns on arbitrary **marginal operators** $\lambda^i \mathcal{O}_i$ which correspond to **massless fields** in space time.

In this way one explores a (consistent) subset of the space of all 2-d conformal field theories.

The β-functions for the coupling constants λ correspond to the classical equations of motion.

In some restricted cases, it is even possible to write down an action which generates these equations of motion in certain classes of 2-d Field theories. [Witten]

Unfortunately, however, beyond the **marginal operators**, one has to also impose a **regularization** scheme for the space of 2-d field theories.

Such a scheme typically **explicitly breaks the conformal invariance on the worldsheet**, which is somewhat difficult to reconcile with the equations of motion which are supposed to be the conditions for conformality.

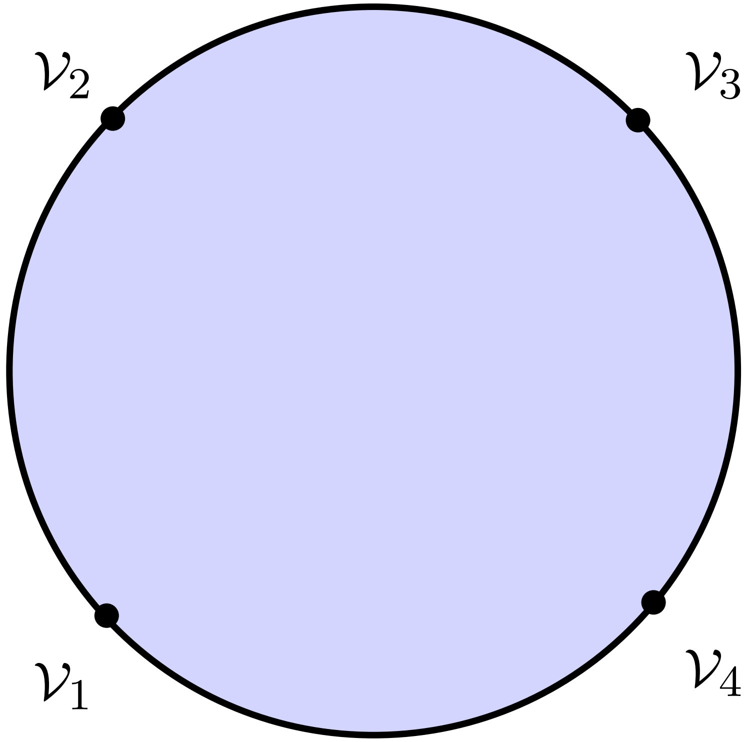
This makes it very **ambiguous** what one means by "the space of 2-d field theories" and scant progress has been made extending such actions to the **massive string fields**.

String Field Theories

There is a completely different approach, which dates from the eariest days of string theory and its quantization in light-cone gauge [Mandlestam].

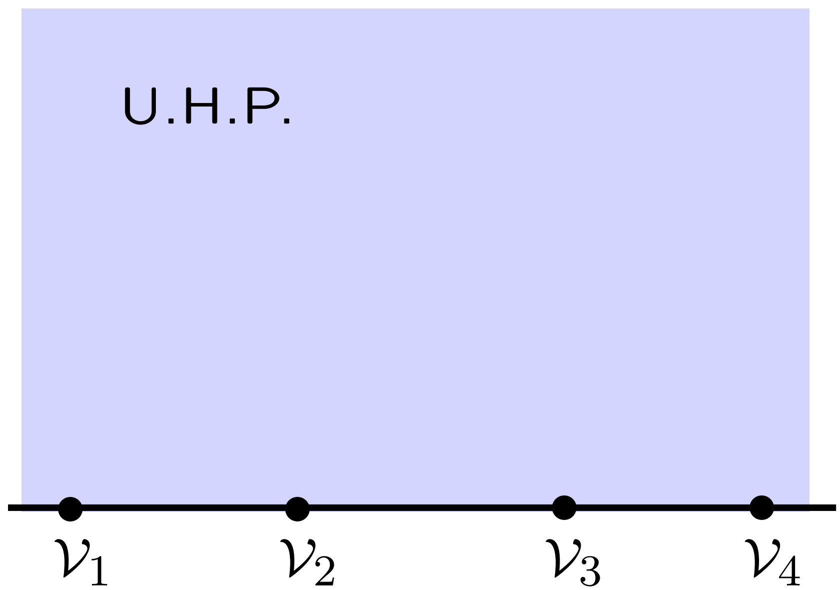
The idea is to consider a very special conformal frame in which the world-sheet diagrams look like Feynman diagrams [Witten; Zwiebach]...

Consider a tree level open string scattering amplitude



Since the string worldsheet action is conformally invariant, we can apply an arbitrary conformal transformation to the disk to yield another shape.

For example, it is common to map the disk to the upper-half plane.



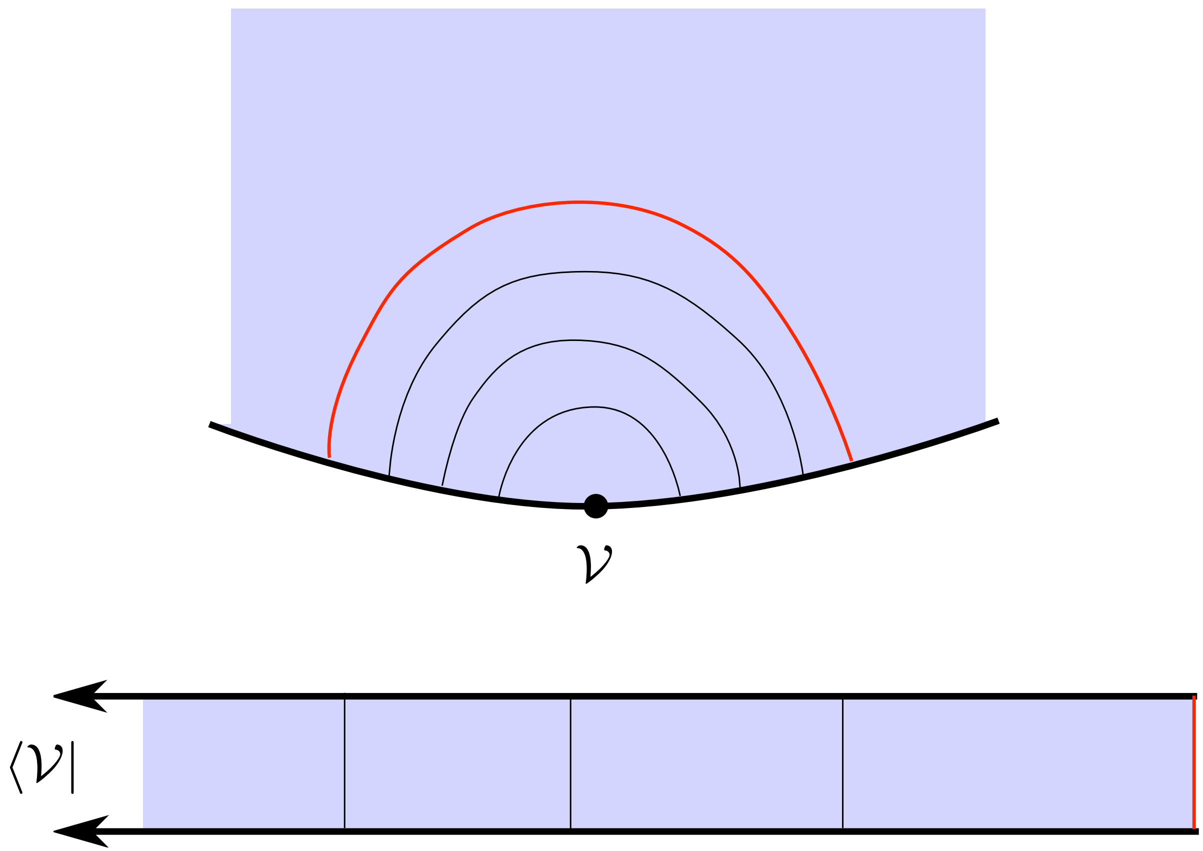
For SFT we require a rather exotic conformal frame [Witten; Giddings, Martinec, Witten; Zwiebach]

This frame can be found by finding a **minimal area** metric on the string world sheet subject to the constraint that every non-contractable Jordan open curve be of at least length π.

Note that without the constraint on the curve lengths, the minimal area would be zero.

What do these metrics look like?

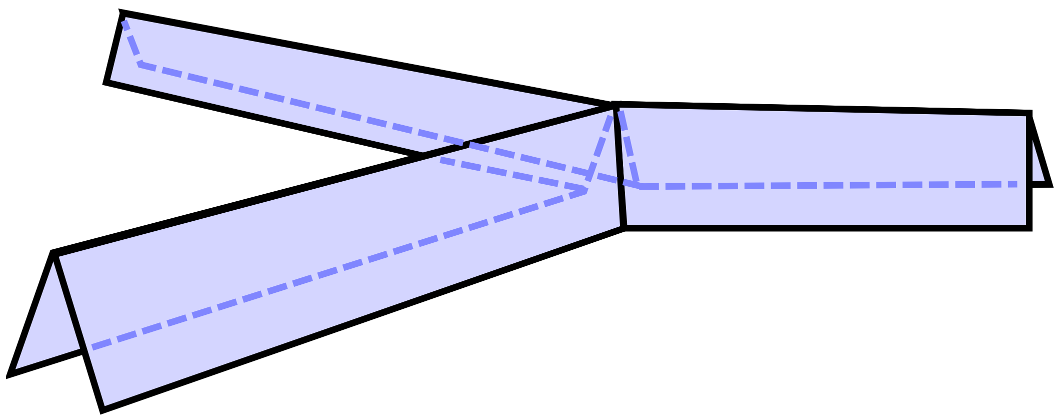
The regions of worldsheet around punctures are mapped via $w \simeq \log(z)$ to semi-infinite strips:



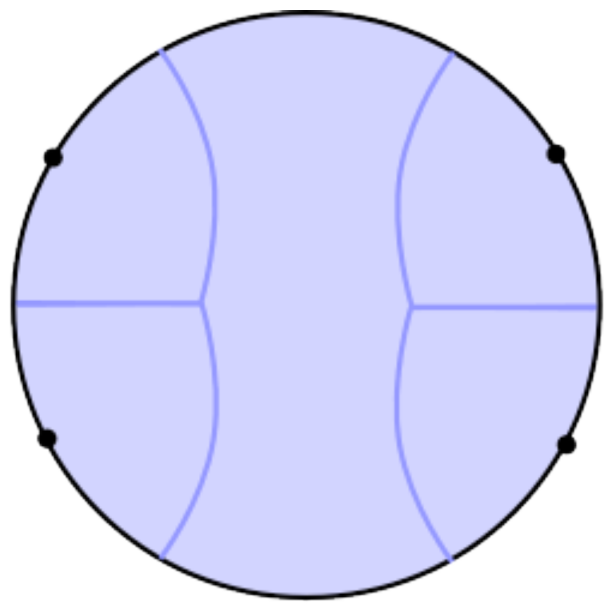
The rest of the world sheet is formed from **two ingredients**

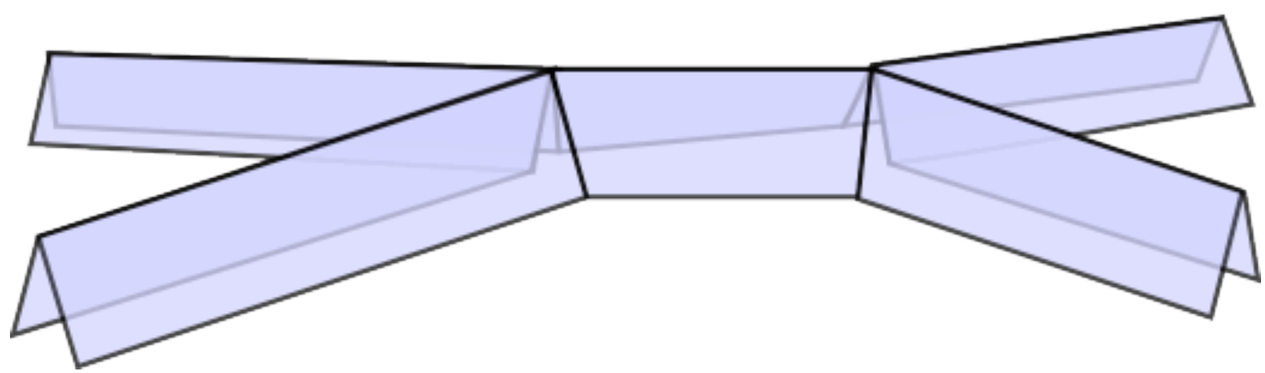
Propagator

Vertex



Example: Veneziano amplitude





What action generates these diagrams?

It is convenient to denote the three vertex as

$$\langle \mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3 \rangle$$

We also denote the **BPZ inner product** (i.e. the disk 2-point function) as

$$\langle \mathcal{V}_1 | \mathcal{V}_2 \rangle$$

Then we have the action

$$S(\Psi) = \frac{1}{2}\langle \Psi | Q_B \Psi \rangle + \frac{g}{3}\langle \Psi, \Psi, \Psi \rangle$$

where Ψ, our "string field" is an arbitrary linear combination of the possible (ghost number 1) vertex operators

$$\Psi = \sum \lambda^i \mathcal{V}_i$$

For example, for the open string field theory on a D25-brane, we have

$$\Psi = \int d^{25}p \left\{ t(p) c_1 e^{ipX} + A_\mu(p) \partial X^\mu c_1 e^{ipX} + \psi(p) c_0 e^{ipX} + \ldots \right\}$$

It should be clear how the cubic term will generate the correct cubic vertex, but how does the kinetic term generate the correct propagator?

Since $Q_B^2 = 0$, the free action has a gauge invariance

$$\mathcal{V} \to \mathcal{V} + Q_B \mathcal{V} \tag{2}$$

A standard choice for fixing this gauge invariance is to pick $b_0 \mathcal{V} = 0$.

For such $\mathcal{V}$, we have

$$\langle \mathcal{V} | Q_B | \mathcal{V} \rangle = \langle \mathcal{V} | c_0 L_0 | \mathcal{V} \rangle$$

The propagator is then the inverse of this, which works out to be

$$D(\mathcal{V}_1, \mathcal{V}_2) = \langle \mathcal{V}_1 | \frac{b_0}{L_0} | \mathcal{V}_2 \rangle \tag{3}$$

However, using the Schwinger representation,

$$b_0 L_0^{-1} = b_0 \int_0^\infty dT e^{-T L_0}$$

And recalling that L_0 is just the world sheet Hamiltonian, we see that the propagator is nothing but an integral over strips of length T.

The star-product

We can define a **star-product** through the relation,

$$\langle \mathcal{V}_1 | \mathcal{V}_2 * \mathcal{V}_3 \rangle = \langle \mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3 \rangle$$

This product is associative

$$\mathcal{V}_1 * (\mathcal{V}_2 * \mathcal{V}_3) = (\mathcal{V}_1 * \mathcal{V}_2) * \mathcal{V}_3 \tag{4}$$

but noncommutative!

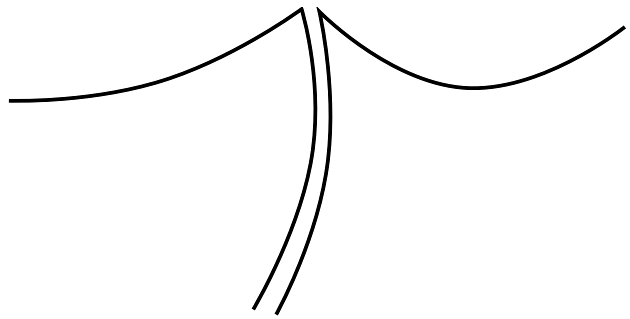
$$\mathcal{V}_2 * \mathcal{V}_2 \neq \mathcal{V}_2 * \mathcal{V}_1 \qquad \text{(in general)} \tag{5}$$

It is also useful to define a notion of string "integration" through

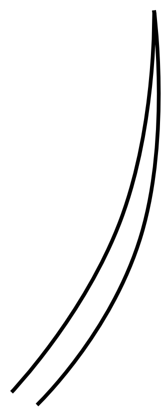
$$\langle \mathcal{V}_1 | \mathcal{V}_2 \rangle = \int \mathcal{V}_1 * \mathcal{V}_2 \tag{6}$$

The $*$-product and integration have a simple geometric interpretation:

$*$



$\int$



Using these two definitions, the cubic vertex takes the form

$$\langle \mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3 \rangle = \int \mathcal{V}_1 * \mathcal{V}_2 * \mathcal{V}_3 \tag{7}$$

We can then write for the action as

$$S(\Psi) = \frac{1}{2} \int \Psi * Q_B \Psi + \frac{g}{3} \int \Psi * \Psi * \Psi \tag{8}$$

Define the graded commutator of two string fields by

$$[A, B] = A * B - (-1)^{gh(A)gh(B)} B * A \tag{9}$$

It is straightforward to check, then that $S(\Psi)$ is invariant under the infinitesimal gauge transformation

$$\Psi \rightarrow \Psi + Q_B \Lambda + g[\Psi, \Lambda]$$

which is analogous to the gauge invariance of non-abelian Yang-Mills.

Classical Vacua

Consider first the **free** theory, $g = 0$.

$$S(\Psi)_{\text{free}} = \frac{1}{2} \int \Psi * Q_B \Psi$$

The equations of motion are

$$Q_B \Psi = 0$$

So the **classical solutions** are the **BRST closed states**.

We also have the gauge invariance,

$$\Psi \to \Psi + Q_B \Lambda$$

So the **physical states** are the Q_B-closed states modulo the Q_B-exact states.

Such states are said to live in the **cohomology** of Q_B.

As is standard (see for example Polchinski Ch. 4) the cohomology of Q_B corresponds to the **spectrum of free string theory**.

Classical vacua

Now consider the case $g \neq 0$. The equations of motion take the form

$$Q_B \Psi + g\Psi * \Psi = 0$$

Rescaling $\Psi \to g^{-1}\Psi$, we can eliminate the factor of g to give just

$$Q_B \Psi + \Psi * \Psi = 0$$

This equation is (secretly) an **infinite number** of **coupled non-linear differential equations** and there is no known way to solve it in general.

Suppose, however, we have a solution Ψ_{cl}. We can then compute

$$S(\Psi + \Psi_{\text{cl}}) = \frac{1}{2}\int \Psi * Q_{\Psi_{\text{cl}}} \Psi + \frac{1}{3}\int \Psi * \Psi * \Psi + S(\Psi_{\text{cl}})$$

Note that it is quite remarkable that the form of the action is unchanged except for the replacement

$$Q_B \to Q_{\Psi_{\text{cl}}} = Q_B + [\Psi_{\text{cl}}, \quad]\ .$$

One can check that $Q^2_{\Psi_{\text{cl}}} = 0$ using the equations of motion.

Classical Vacua

Hence, a classical solution of OSFT is characterized by two pieces of data

1. A new BRST operator $Q_{\Psi_{\mathsf{cl}}}$ whose cohomology determines the spectrum around the new vacuum.

2. The constant $S(\Psi_{\mathsf{cl}})$ which, if Ψ_{cl} is independent of time, is just the difference in energy between the old vacuum and the new vacuum.

So, are there any interesting solutions to the OSFT equations of motion?

For one special case, the answer was yes:

Suppose one has a **truly marginal operator** $\lambda\mathcal{O}$ in some CFT which **deforms** the CFT to a CFT'(λ).

It was shown by Sen and Zwiebach that one can always find a corresponding string field $\Psi(\lambda)$ which solves the full non-linear equations of motion.

This was only a statement about **existence**, however. It is still not known how to construct such solutions explicitly in general.

Tachyon Condensation

It is quite remarkable that in 1989 Kostelecky and Samuel demonstrated the existence of a non-trivial solution to the equations of motion which is not of the kind described by Sen and Zwiebach.

Their method was as follows:

The bosonic open string field contains as it's first component a tachyon field t.

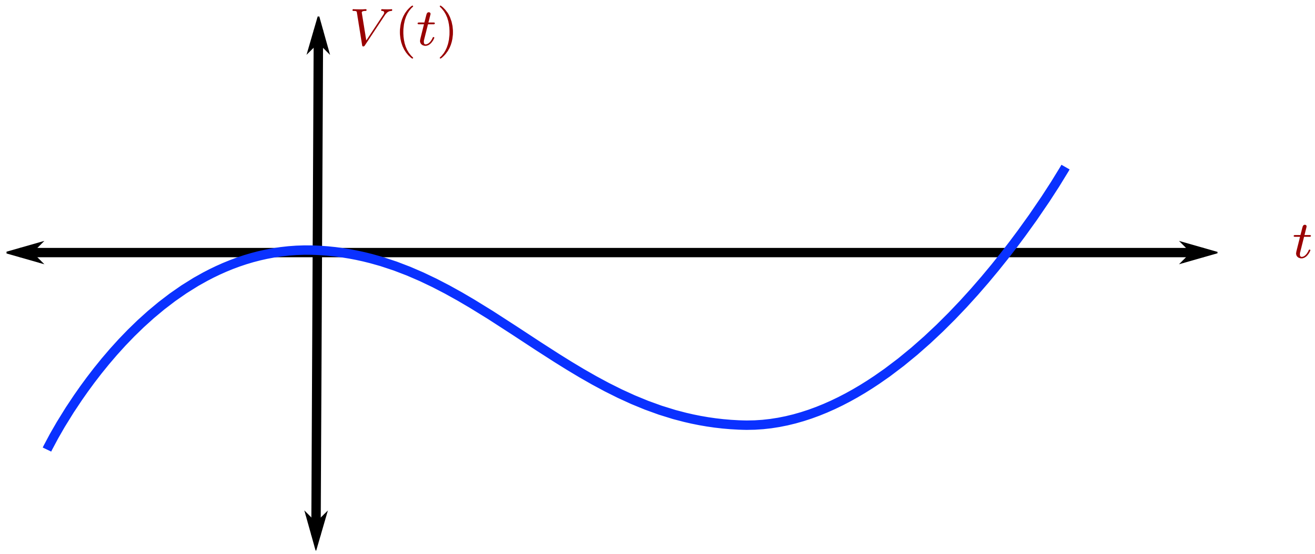
Fixing this field to be a constant one can use the OSFT equations of motion to integrate out the other fields giving a string field $\Psi(t)$.

Plugging this back into the action gives an effective potential for the tachyon

$$V(t) = S(\Psi(t))$$

To make the computation possible numerically, they truncated the number of fields by throwing out fields above some fixed mass, a procedure they called **level truncation**.

They found something like this:



Amazingly, there appeared to be a minimum of the tachyon potential corresponding to a new solution.

Unfortunately, it would be 10 years before the interpretation of this vacuum was clear.

Sen's Conjectures

The proper interpretation of this vacuum was given in 1999 by Sen.

Open string theory can be understood as capturing the **dynamics of D-branes**.

An **open string tachyon** should be interpreted as an **instability** of a particular D-brane configuration to decay.

The tachyon vacuum, then, should be interpreted as the configuration in which the D-brane that the open strings ended on has decayed and hence,

Conjecture 1: The difference in energy between the perturbative vacuum and the tachyon vacuum should be the energy (or more accurately the tension) of the orginal D-brane.

Equivalently, $S(\Psi) = -T$.

Furthermore, since in the tachyon vacuum there is no brane, **there should be no open strings**.

But, since there are no open strings, there should be **no perturbative excitations around the tachyon vacuum**. Hence,

Conjecture 2: The cohomology of Q_Ψ should vanish.

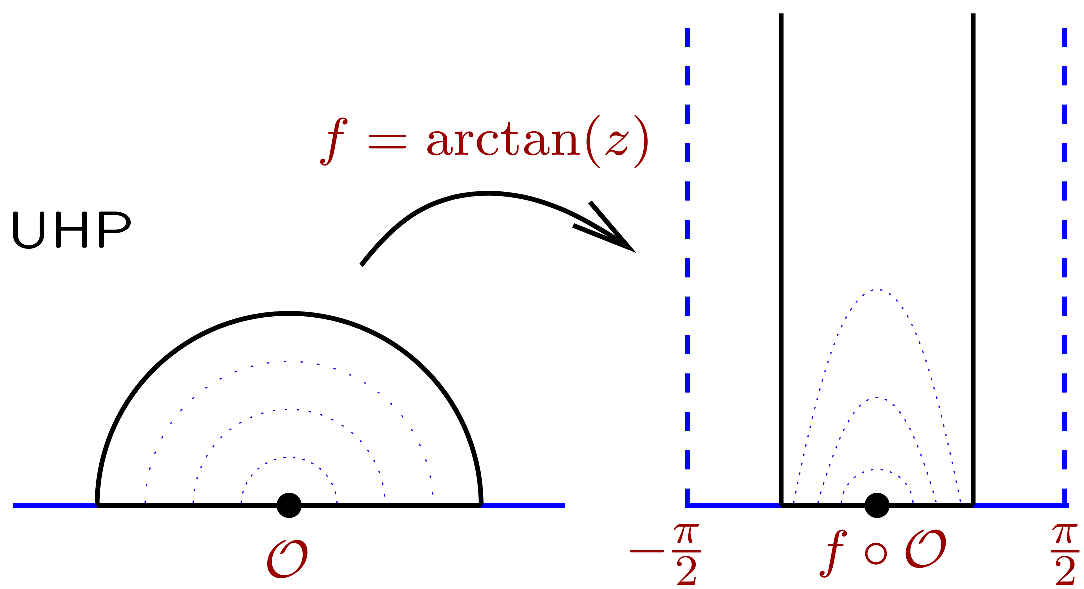
These conjectures were tested with great precision numerically, by many people [Sen,Zwiebach; Moeller, Sen, Zwiebach; Moeller, Taylor; Gaiotto, Rastelli; IE, Taylor; IE, Feng, He, Moeller]

In this talk I will mainly focus on the recent analytic proofs of them.

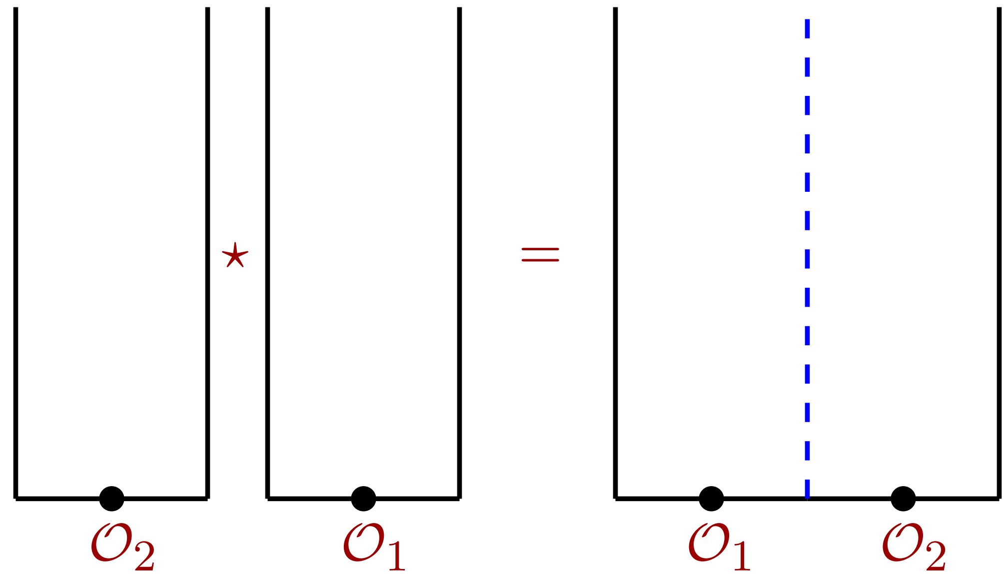
The Schnabl solution

Recently, M. Schnabl found an exact solution to the OSFT equations of motion

To describe the solution it is very useful to introduce a new coordinate system in which the star product simplifies.



In these coordinates, the star product takes a very simple form



Note that there are **no conformal transformations** applied to the $\mathcal{O}$'s.

Because of this simplicity, objects defined in the $\arctan(z)$ coordinates tend to have nice properties under star multiplication.

Given an object in $\arctan(z)$ coordinates, we can pull it back to the original coordinates.

For example, the Hamiltonian L_0 can be pulled back to give

$$\mathcal{L}_0 \equiv f^{-1} \circ L_0 = L_0 + \tfrac{2}{3} L_2 - \tfrac{2}{15} L_4 + \ldots$$

This operator has many beautiful properties.

It's BPZ conjugate, $\mathcal{L}_0^\star$ is given by

$$\mathcal{L}_0^\star = L_0 + \tfrac{2}{3} L_{-2} - \tfrac{2}{15} L_{-4} + \ldots$$

And the two operators satisfy the simplest Lie algebra,

$$[\mathcal{L}_0, \mathcal{L}_0^\star] = \mathcal{L}_0 + \mathcal{L}_0^\star \ .$$

Using this relation, it is easy to check that $\widehat{\mathcal{L}} = \mathcal{L}_0 + \mathcal{L}_0^\star$ raises the $\mathcal{L}_0$ eigenvalue by one, since

$$[\mathcal{L}_0, \widehat{\mathcal{L}}] = \widehat{\mathcal{L}} \ .$$

One can also show the following identities

$$(\widehat{\mathcal{L}}\phi_1) * \phi_2 = \widehat{\mathcal{L}}(\phi_1 * \phi_2) + \tfrac{\pi}{2}\phi_1 * (K_1\phi_2)$$
$$\phi_1 * (\widehat{\mathcal{L}}\phi_2) = \widehat{\mathcal{L}}(\phi_1 * \phi_2) - \tfrac{\pi}{2}(K_1\phi_1) * \phi_2$$

where $K_1 = L_1 + L_{-1}$.

These allow one to star multiply fields with arbitrary numbers of $\widehat{\mathcal{L}}$'s.

To complete the story, one also adds the new operator,

$$\mathcal{B}_0 = f^{-1} \circ b_0$$

and

$$\widehat{\mathcal{B}} = \mathcal{B}_0 + \mathcal{B}_0^\star$$

which satisfies similar identities to $\widehat{\mathcal{L}}$, and one defines

$$\tilde{c}_m = f^{-1} \circ c_m.$$

which have eigenvalue $-m$ with respect to $\mathcal{L}_0$.

It was Schnabl's important discovery that the ghost number 1 states formed from these operators,

$$\Psi = f_{n,p}\,\widehat{\mathcal{L}}^n \tilde{c}_p|0\rangle + f_{n,p,q}\,\widehat{\mathcal{B}}\,\widehat{\mathcal{L}}^n \tilde{c}_p \tilde{c}_q|0\rangle$$

form a **closed algebra** under star multiplication.

Even more importantly, he showed that if ϕ_1 has $\mathcal{L}_0$ eigenvalue h_1 and ϕ_2 has $\mathcal{L}_0$ eigenvalue h_2 then $\phi_1 * \phi_2$ is a sum of states with eigenvalue $h \geq h_1 + h_2$.

The important implication of this is that the star products of low level fields are **not affected by higher level fields**, so one can solve for the coefficients $f_{n,p}$ and $f_{n,p,q}$ exactly.

Solving for the coefficients up to high level, Schnabl was able to guess the complete solution,

$$\Psi = \sum_{n=0}^{\infty} \sum_{p \text{ odd}} \left(\frac{\pi^p (-1)^n}{2^{n+2p+1} n!} \right) B_{n+p+1} \widehat{\mathcal{L}}^n \tilde{c}_{-p} |0\rangle$$

$$+ \sum_{n=0}^{\infty} \sum_{p+q \text{ odd}} \left(\frac{\pi^{p+q} (-1)^{n+q}}{2^{n+2(p+q)+3} n!} \right) B_{n+p+q+1} \widehat{\mathcal{B}}\, \widehat{\mathcal{L}}^n \tilde{c}_{-p} \tilde{c}_{-q} |0\rangle$$

where B_n is the nth Bernoulli number.

This state has a much simpler description which one can find using the Euler-Maclaurin series

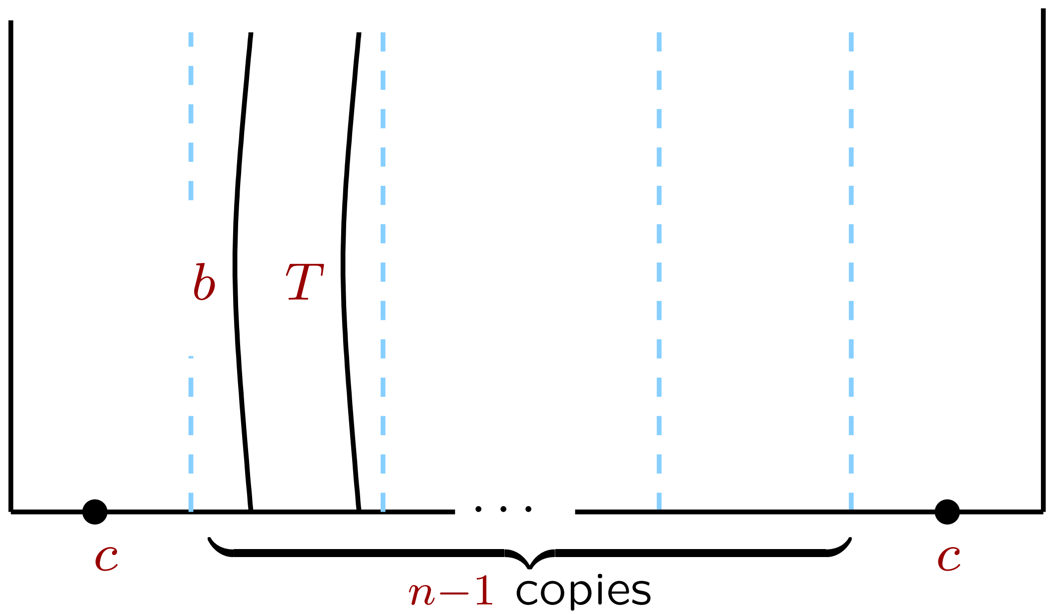
$$\sum_{n=0}^{\infty} \frac{B_n}{n!} \left[f^{(n)}(b) - f^{(n)}(a) \right] = \sum_{k=a}^{b-1} f'(k)$$

Schnabl's solution is then given by

$$\Psi = \sum_{n=0}^{\infty} \chi_n$$

where

$\chi_n =$



The state Ψ satisfies a gauge fixing condition which is the analogue of the Feynman-Siegal gauge mentioned earlier.

$$\mathcal{B}_0 \Psi = 0$$

where

$$\mathcal{B}_0 = f^{-1} \circ b_0$$

Unfortunately, the proof that the **energy** of this state comes out correctly is still very complex, although it boils down to just plugging the solution in the action and being careful about the ordering of certain sums.

Since the cubic term and the kinetic term are related by the equations of motion, one can in fact compute them separately to check the energy in two different ways. [Schnabl, Okawa]

Nonetheless, at this point, we have a more of less complete proof of Sen's first conjecture.

Vanishing cohomology

In general, finding the cohomology of Q_Ψ could be very challenging. However, when there is **no** cohomology, there is a trick [IE, Feng, He,Moeller]

There exists an **identity** string field, $\mathcal{I}$, which satisfies

$$\mathcal{I} \star \Lambda = \Lambda$$

for suitably **well-behaved** states Λ.

Theorem: If there exists a state A such that

$$Q_\Psi A = Q_B A + \Psi \star A + A \star \Psi = \mathcal{I}$$

then the **cohomology** of Q_Ψ **vanishes** (up to certain caveats).

Proof:

Suppose $Q_\Psi \Lambda = 0$. Then

$$Q_\Psi (A \star \Lambda) = Q_\Psi A \star \Lambda = \mathcal{I} \star \Lambda = \Lambda$$

Hence, Λ is exact.

An ansatz for A

For the **Feynman-Siegal gauge** solution it was possible to find the A state **numerically** [IE, Feng, He, Moeller]

The result was consistent with

$$A \simeq \frac{1}{L_0} b_0 \mathcal{I}$$

Mysteriously, this field is as close as one can get to finding a field A in the perturbative vacuum!

$$Q_B A = \mathcal{I} - |0\rangle.$$

The corresponding field for the Schnabl solution is obvious to guess

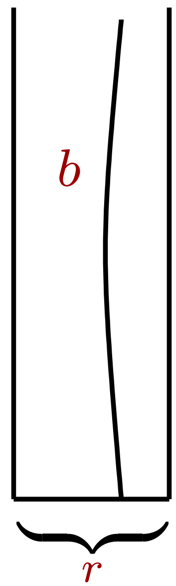
$$A = \frac{1}{\mathcal{L}_0} \mathcal{B}_0 \mathcal{I}$$

where

$$\mathcal{L}_0 = f^{-1} \circ L_0 \qquad \mathcal{B}_0 = f^{-1} \circ b_0$$

The state A has a nice **geometric form** in the $\arctan(z)$ coordinate

$$A = -\int_0^{\pi/2} dr$$



Note that acting on A with Q_B simply replaces the contour integral of $b(z)$ by a contour integral of $T(z)$.

However, an insertion of $\int dz\, T(z)$ is equivalent to a derivative with respect to the width of the strip, r so we only get contributions from the boundary of the integral.

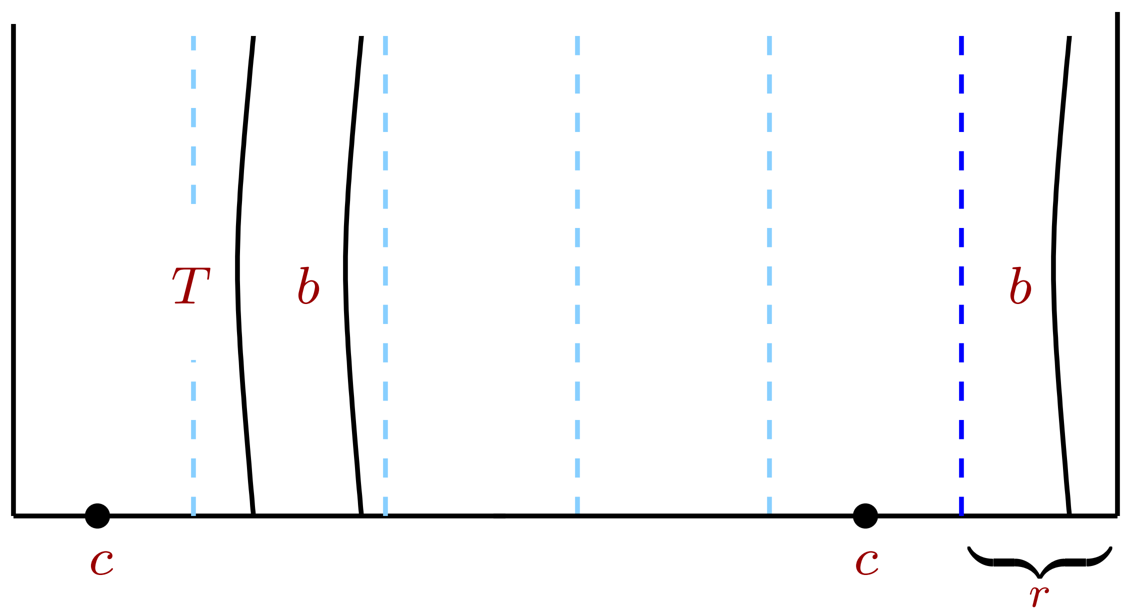
$$Q_B A = \mathcal{I} - |0\rangle$$

To complete the discussion, we need to compute $\Psi \star A$ and $A \star \Psi$.

Consider $A \star \chi_n$

Pictorially:

$$-\int_0^{\pi/2} dr$$



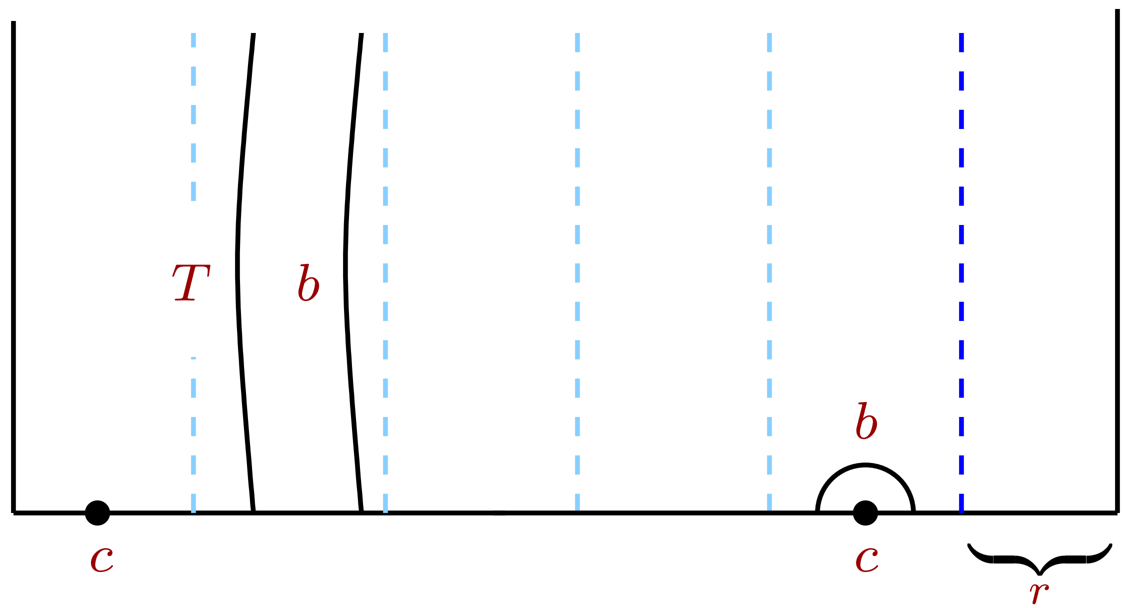
Pull the b contour on the far right to the left. Since $(\int b)^2 = 0$, we just get a contribution from the b contour circling the c insertion.

To complete the discussion, we need to compute $\Psi \star A$ and $A \star \Psi$.

Consider $A \star \chi_n$

Pictorially:

$$-\int_0^{\pi/2} dr$$



Now

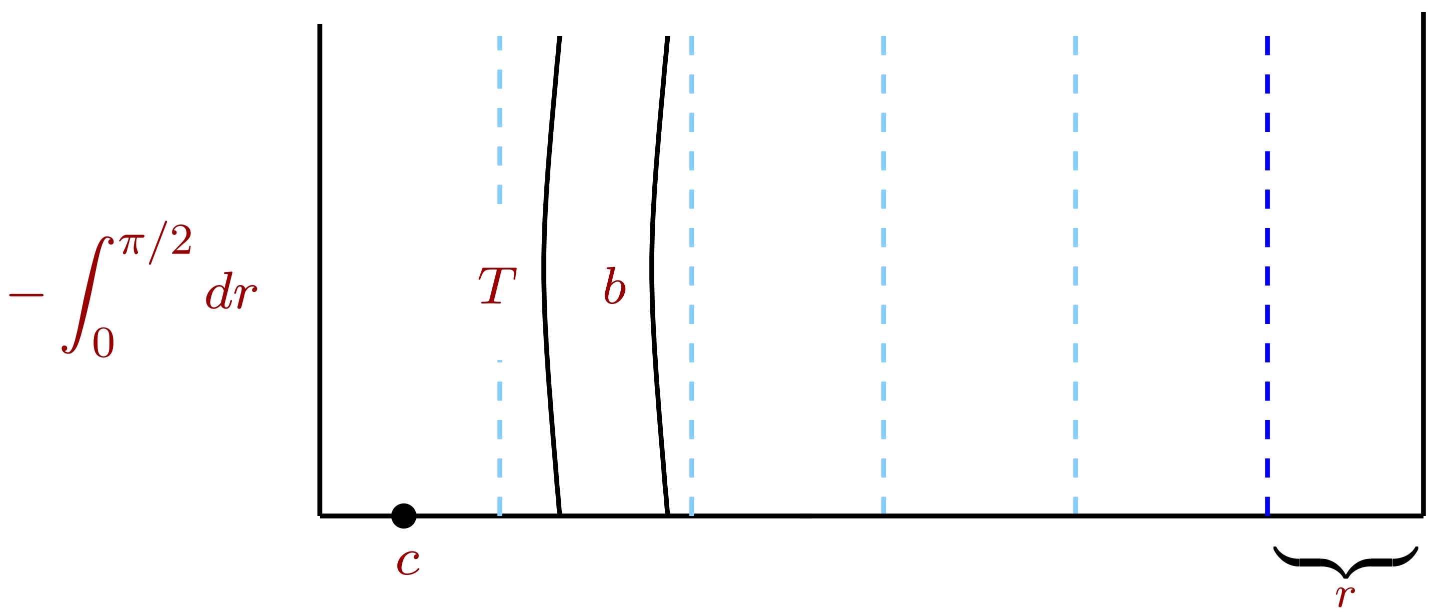
$$\oint dz\, b(z)\; c(w) = 1$$

So we are left with. . .

To complete the discussion, we need to compute $\Psi \star A$ and $A \star \Psi$.

Consider $A \star \chi_n$

Pictorially:



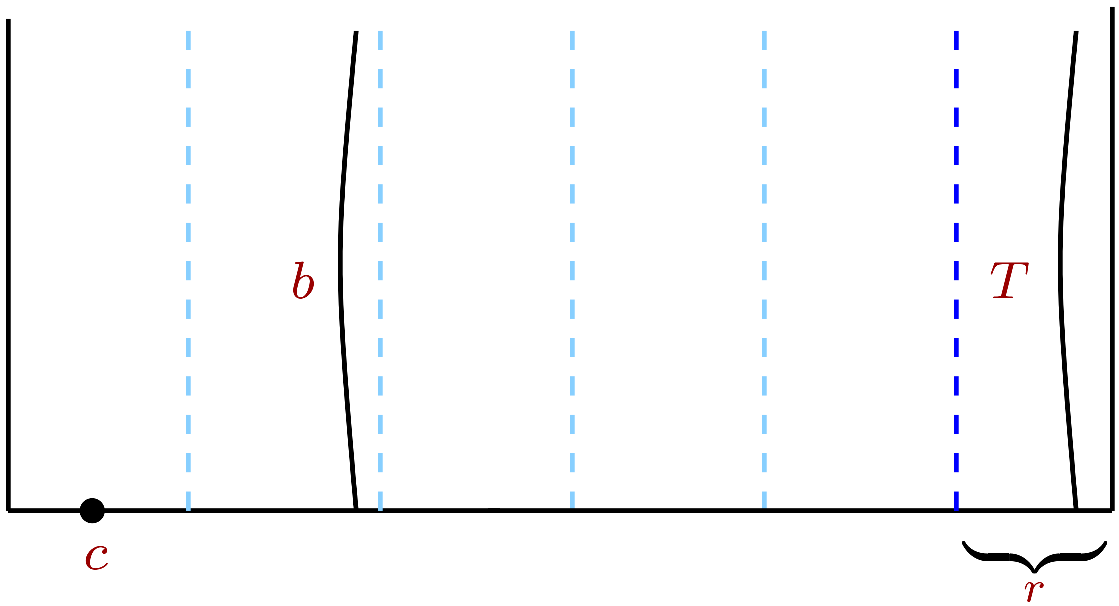
Now pull the T contour to the right

To complete the discussion, we need to compute $\Psi \star A$ and $A \star \Psi$.

Consider $A \star \chi_n$

Pictorially:

$$-\int_0^{\pi/2} dr$$



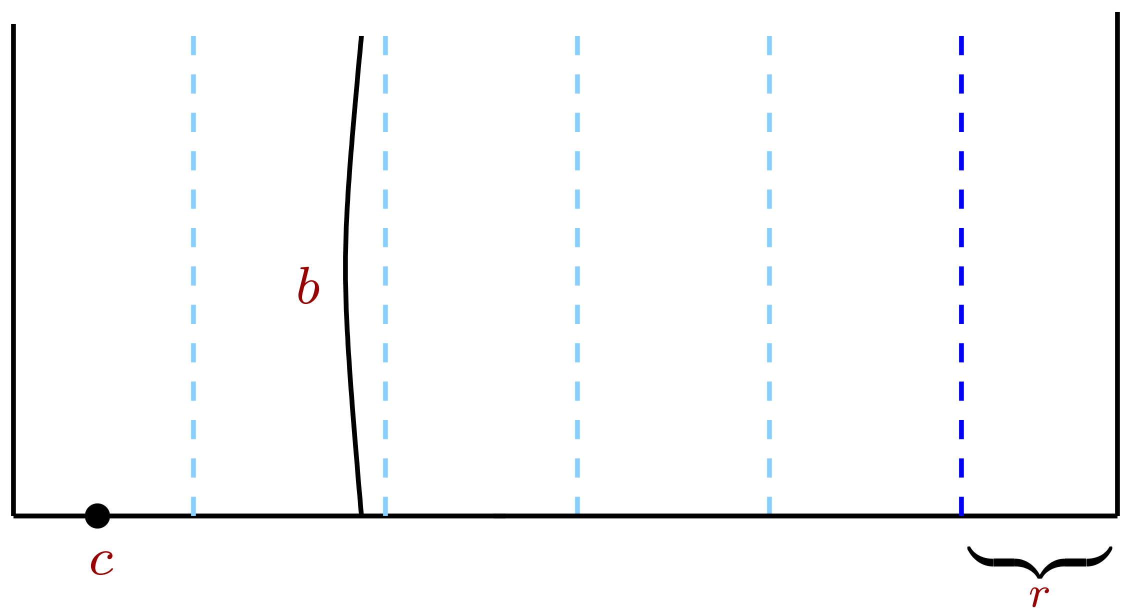
By the same argument we used before, we can replace the T contour by a derivative w.r.t r

To complete the discussion, we need to compute $\Psi \star A$ and $A \star \Psi$.

Consider $A \star \chi_n$

Pictorially:

$$-\int_0^{\pi/2} dr\, \frac{\partial}{\partial r}$$

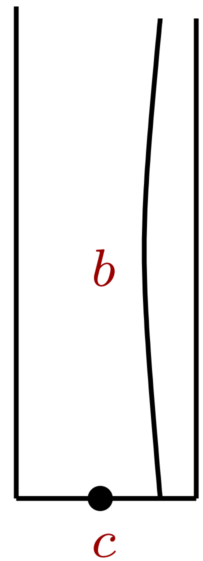


So we only get contributions at $r = 0$ and $r = \pi/2$

Summing over n, everything cancels except a piece from the $n = 0$ term.

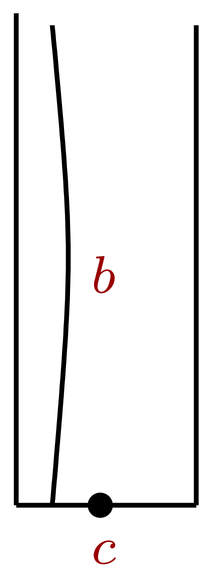
Hence we find the simple result

$$A \star \Psi =$$



One can also compute

$$-\Psi \star A =$$



Adding, we find $A \star \Psi + \Psi \star A = |0\rangle$.

All together, we have discovered that

$$\begin{aligned} Q_\Psi A &= Q_B A + A \star \Psi + \Psi \star A \\ &= \mathcal{I} - |0\rangle + |0\rangle \\ &= \mathcal{I} \end{aligned}$$

This gives a simple proof of **Conjecture 2**.

Can we climb the tachyon hill?

Most of this talk has been about how one can find the minimum of the tachyon potential and reduce the number of branes we have by condensing some of them.

One can ask whether one can also climb **up** the tachyon potential and find states with **more branes than we started with**.

Unlike with the tachyon vacuum we have no numerical solutions to suggest that this is possible.

However, perhaps we can use the form of the tachyon solution to **guess** a solution for multiple branes.

Pure gauge form

One of the remarkable things about Martin's solution is that it can also be written as [Okawa]

$$\Psi = \lim_{\lambda \to 1} U * (Q_B V)$$

where

$$U = 1 - \lambda\Phi \ , \qquad V = \frac{1}{1 - \lambda\Phi}$$

and

$$\Phi = B_1^L c_1 |0\rangle$$

Note that for $\lambda < 1$ the state $U * (Q_B V)$ takes the form of a pure gauge state since $V = U^{-1}$.

(This is analogous to a gauge field of the form $A_\mu = g^{-1}\partial_\mu g$)

At exactly $\lambda = 1$, however, there exists a state $|\infty\rangle$ (the wedge state of infinite width) that is annihilated by U

$$U * |\infty\rangle = 0 \qquad (\text{at } \lambda = 1) \ .$$

In fact, one also has

$$|\infty\rangle * U = |\infty\rangle \qquad |\infty\rangle * |\infty\rangle = |\infty\rangle$$

So these operators generate an associativity anomaly!

$$|\infty\rangle * (U * |\infty\rangle) = 0 \ ,$$
$$(|\infty\rangle * U) * |\infty\rangle = |\infty\rangle \ .$$

This is only one of many anomalies associated with the state $|\infty\rangle$.

A Guess for the 2-brane state (with M. Schnabl)

To construct a **multiple brane solution**, consider the form of the perturbative vacuum from the perspective of the tachyon vacuum:

$$-\Psi = -U * Q_B V = V * Q_\Psi U$$

We guess, then, that the **2-brane solution** will take the form

$$V^2 * Q_\Psi U^2 \ .$$

Around the perturbative vacuum, this reduces to the state,

$$\Psi_2 = V * Q_B U \ .$$

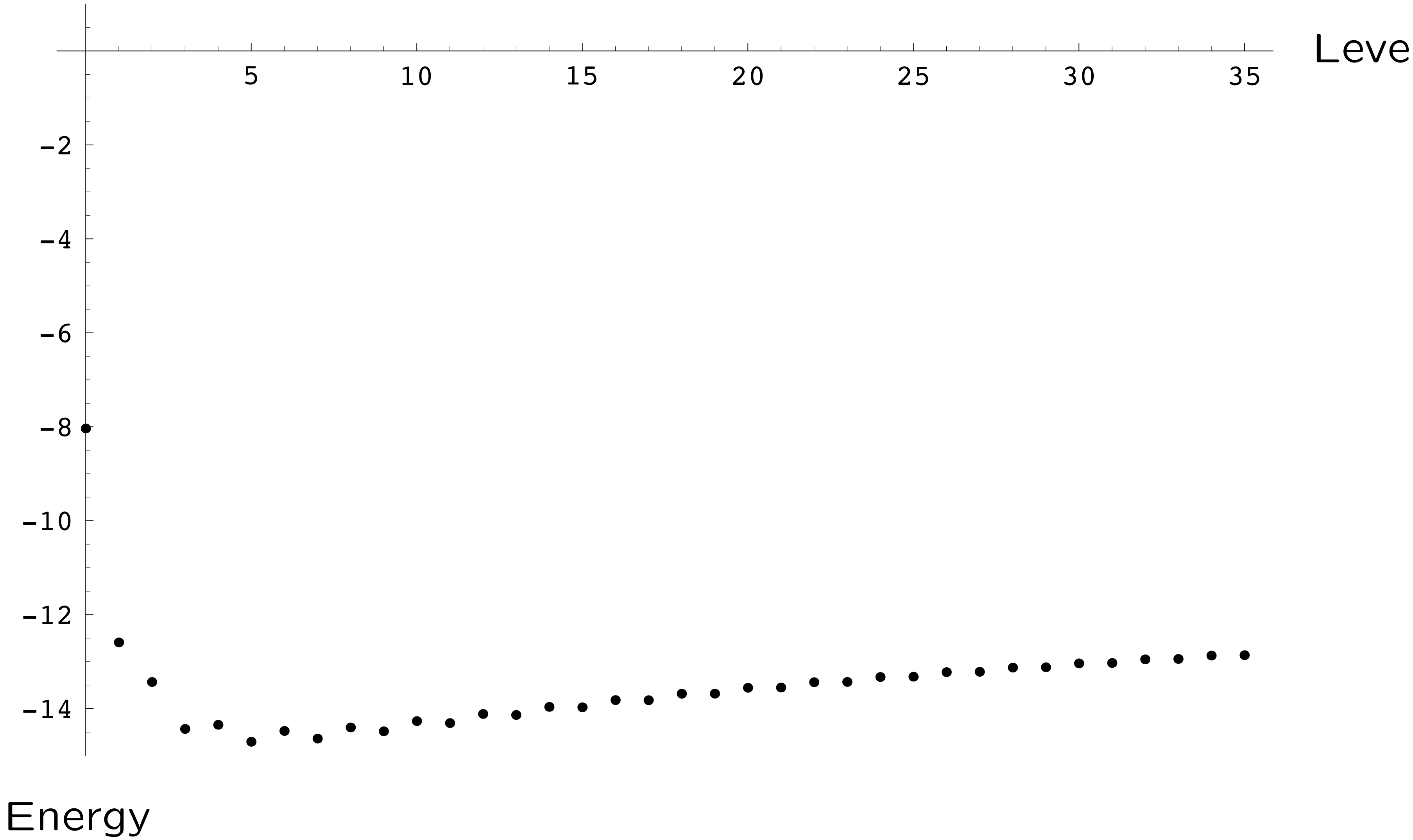
Formally, this state has precisely **minus the energy of the tachyon vacuum** as can be seen by computing the cubic term,

$$\int \Psi_2 * \Psi_2 * \Psi_2 = \int V * Q_B U * V * Q_B U * V * Q_B U \ .$$

Moving the Q_B's from the U's to the$V$'s using that $Q_B(V * U) = 0$ gives

$$-\int U * Q_B V * U * Q_B V * U * Q_B V = -\int \Psi * \Psi * \Psi \ .$$

Do these formal considerations work in ordinary level truncation?



All we can say from this is that the energy is **headed in the right direction**.

We are currently trying to check whether the **equations of motion** are satisfied in level truncation and in addition, to check that the **cubic tems** yield reasonable energies.

At this time we can say that there is some hope that this is an honest solution of OSFT, but we do not have any definitive checks as yet.

The End!