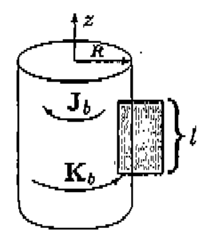
PHY 322 Final Exam Solutions. (May 12, 2004)

Problem 1

(a) $\oint \mathbf{B}\cdot d\mathbf{l} = B\,2\pi s = \mu_0 I_{enc} \Rightarrow \boxed{\mathbf{B} = \left\{\begin{array}{ll} 0, & \text{for } s < a; \\ \dfrac{\mu_0 I}{2\pi s}\,\hat{\boldsymbol{\phi}}, & \text{for } s > a. \end{array}\right\}}$

(b) $J = ks$; $I = \int_0^a J\,da = \int_0^a ks(2\pi s)\,ds = \dfrac{2\pi k a^3}{3} \Rightarrow k = \dfrac{3I}{2\pi a^3}$. $I_{enc} = \int_0^s J\,da = \int_0^s k\bar{s}(2\pi\bar{s})\,d\bar{s} = \dfrac{2\pi k s^3}{3} = I\dfrac{s^3}{a^3}$, for $s < a$; $I_{enc} = I$, for $s > a$. So $\boxed{\mathbf{B} = \left\{\begin{array}{ll} \dfrac{\mu_0 I s^2}{2\pi a^3}\,\hat{\boldsymbol{\phi}}, & \text{for } s < a; \\ \dfrac{\mu_0 I}{2\pi s}\,\hat{\boldsymbol{\phi}}, & \text{for } s > a. \end{array}\right\}}$

Problem 2



(a) $\mathbf{M} = ks\hat{\mathbf{z}}$; $\mathbf{J}_b = \nabla\times\mathbf{M} = -k\hat{\boldsymbol{\phi}}$; $\mathbf{K}_b = \mathbf{M}\times\hat{\mathbf{n}} = kR\hat{\boldsymbol{\phi}}$.

B is in the z direction (this is essentially a superposition of solenoids). So $\boxed{\mathbf{B} = 0 \text{ outside.}}$ Use the amperian loop shown (shaded)—inner side at radius s:

$\oint \mathbf{B}\cdot d\mathbf{l} = Bl = \mu_0 I_{enc} = \mu_0\left[\int J_b\,da + K_b l\right] = \mu_0\left[-kl(R-s) + kRl\right] = \mu_0 kls.$

$\therefore \boxed{\mathbf{B} = \mu_0 ks\hat{\mathbf{z}} \text{ inside.}}$

(b) By symmetry, **H** points in the z direction. That same amperian loop gives $\oint \mathbf{H}\cdot d\mathbf{l} = Hl = \mu_0 I_{f_{enc}} = 0$, since there *is* no free current here. So $\boxed{\mathbf{H} = 0}$, and hence $\boxed{\mathbf{B} = \mu_0\mathbf{M}.}$ *Outside* $\mathbf{M} = 0$, so $\mathbf{B} = 0$; *inside* $\mathbf{M} = ks\hat{\mathbf{z}}$, so $\mathbf{B} = \mu_0 ks\hat{\mathbf{z}}$.

Problem 3

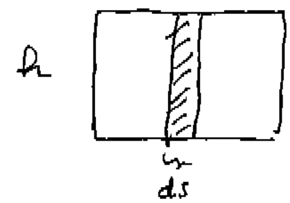
(a) The magnetic field due to $I = I_0 \cos\omega t$ is

$$\vec{B} = \frac{\mu_0 I}{2\pi s}\,\hat{\phi}$$

The flux through a single turn:

$$\Phi_1 = \int \vec{B} \cdot d\vec{a}$$

$$= \frac{\mu_0 I}{2\pi} \int_a^b \frac{h\,ds}{s}$$



$$= \frac{\mu_0 I h}{2\pi} \ln \frac{b}{a}$$

Total flux $= N\Phi_1 = \frac{\mu_0 N h}{2\pi} \ln \frac{b}{a} \; I_0 \cos\omega t$

Induced emf:

$$\boxed{\mathcal{E} = -\frac{d\Phi}{dt} = \frac{\mu_0 N h}{2\pi} \ln\left(\frac{b}{a}\right) I_0 \omega \sin\omega t}$$

(b) Current through the resistor

$$\boxed{I_r = \frac{\mathcal{E}}{R} = \frac{\mu_0 N h}{2\pi R} \ln\left(\frac{b}{a}\right) I_0 \omega \sin\omega t}$$

(c). This was derived in the textbook Example 7.11

Find the self-inductance of a toroidal coil with rectangular cross section (inner radius a, outer radius b, height h), which carries a total of N turns.

Solution: The magnetic field inside the toroid is (Eq. 5.58)

$$B = \frac{\mu_0 N I}{2\pi s}.$$

The flux through a single turn (Fig. 7.33) is

$$\int \mathbf{B} \cdot d\mathbf{a} = \frac{\mu_0 N I}{2\pi} h \int_a^b \frac{1}{s}\, ds = \frac{\mu_0 N I h}{2\pi} \ln\left(\frac{b}{a}\right).$$

The *total* flux is N times this, so the self-inductance (Eq. 7.25) is

$$L = \frac{\mu_0 N^2 h}{2\pi} \ln\left(\frac{b}{a}\right). \qquad (7.27)$$

(d) The back emf:

$$\mathcal{E}_b = -L \frac{dI_r}{dt}$$

$$= -\left(\frac{\mu_0 N^2 h}{2\pi} \ln \frac{b}{a}\right)\left[\frac{\mu_0 N h}{2\pi R} \ln\left(\frac{b}{a}\right) I_0 \omega^2 \cos\omega t\right.$$

$$\boxed{\mathcal{E}_b = -\frac{\mu_0^2 N^3 h^2}{(2\pi)^2 R}\left[\ln\left(\frac{b}{a}\right)\right]^2 I_0 \omega^2 \cos\omega t}$$