4/12/04

Vector Potential

Remember that $\vec{\nabla} \times \vec{E} \Rightarrow \vec{E} = -\vec{\nabla} V$ permits the introduction of scalar potential

Similarly $\vec{\nabla} \cdot \vec{B} \Rightarrow \vec{B} = \vec{\nabla} \times \vec{A}$ vector potential

This form of $\vec{B} = \vec{\nabla} \times \vec{A}$ guarantees that $\vec{\nabla} \cdot \vec{B} = 0$

Note that $\vec{A}$ is not unique (just like $V' \to V + V$)

$$\vec{A}' \to \vec{A} + \vec{\nabla} \lambda$$

This leaves $\vec{\nabla} \times \vec{A}$ unchanged. This is called a <u>gauge transformation</u>

A particularly useful gauge condition

$$\vec{\nabla} \cdot \vec{A} = 0$$ known as Coulomb gauge

is usually chosen.

To prove this is always possible:

Suppose $\vec{\nabla}\cdot\vec{A} \neq 0$, let $\vec{A}' = \vec{A} + \nabla\lambda$

such that $\vec{\nabla}\cdot\vec{A}' = 0 \Rightarrow \nabla^2\lambda = -\vec{\nabla}\cdot\vec{A}$

Poisson eqn with $\frac{\rho}{\epsilon_0} = \vec{\nabla}\cdot\vec{A}$

Soln $\Rightarrow$ $\lambda(\vec{r}) = \frac{1}{4\pi}\int \frac{\vec{\nabla}'\cdot\vec{A}'\,d\tau'}{|\vec{r}-\vec{r}'|}$

In any case, we will therefore assume $\vec{\nabla}\cdot\vec{A} = 0$

The equation for $\vec{A}$ is simplified in Coulomb gauge

Ampere's law. $\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$

$\Rightarrow \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \mu_0\vec{J}$

$\Rightarrow \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} = \mu_0\vec{J}$

analogue of $\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}(\vec{a}\cdot\vec{c}) - (\vec{a}\cdot\vec{b})\vec{c}$

$\Rightarrow \nabla^2\vec{A} = -\mu_0\vec{J}$.

basically 3 Poisson eqn

$$\Rightarrow \begin{cases} \nabla^2 A_x = -\mu_0 J_x \\ \nabla^2 A_y = -\mu_0 J_y \\ \nabla^2 A_z = -\mu_0 J_z \end{cases}$$

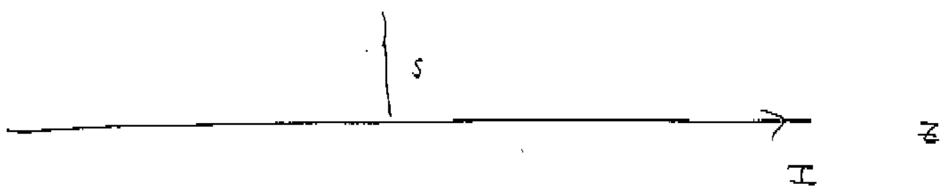
4/14/04

We know the Soln

$$\Rightarrow \quad \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\, d\tau'$$

In a region where $\vec{J}=0$, each component of $\vec{A}$ satisfies the Laplace Eqn. $\Rightarrow$ Machinery in Ch. 3

Again consider the simplest case



$$\vec{B} = \frac{\mu_0 I}{2\pi s}\,\hat{\phi}$$

$$B_\phi = \left(\frac{\partial A_s}{\partial z} - \frac{\partial A_z}{\partial s}\right)$$

For example:

$$\left.\begin{aligned} A_s &= \frac{\mu_0 I z}{2\pi s} \\ A_z &= -\frac{\mu_0 I}{4\pi} \ln s \end{aligned}\right\} \text{can give } B_\phi$$

For Example $A_s = A_\phi = 0$

$$A_z = -\frac{\mu_0 I}{2\pi} \ln s \quad \text{will do}$$

Check: $\vec{\nabla} \times \vec{A} = -\frac{\partial A_z}{\partial s} \hat{\phi} = \frac{\mu_0 I}{2\pi s} \hat{\phi}$ ✓

$$\vec{\nabla} \cdot \vec{A} = \frac{1}{s} \frac{\partial}{\partial s} (s A_s) + \frac{\partial A_z}{\partial z} + \frac{1}{s} \frac{\partial A_\phi}{\partial \phi}$$

$= 0$ satisfies Coulomb gauge

Note that $\vec{A}$ points in the same direction as the current. This is generally true.

If all current points in same direction

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} d\tau' \quad \text{suggest that } \vec{A} \parallel \vec{J}$$

But :

① We can add a constant vector to change direction of $\vec{A}$ without affecting $\vec{B}$

② We can add even a more general $\lambda(\vec{r})$ (gauge transformation). Even without Coulomb gauge, we can choose any $\lambda(\vec{r})$ such that $\nabla^2 \lambda = 0$

Alternatively, $\nabla^2 A_z = 0$ outside wire

$\Rightarrow \quad A_z = c \ln s$ (since the problem has symmetry in ϕ, we do not get $\sin n\phi$ & $\cos n\phi$ dependence.

Boundary condition

$$2\pi s \, (\vec{\nabla} \times \vec{A})_\phi = \mu_0 I$$

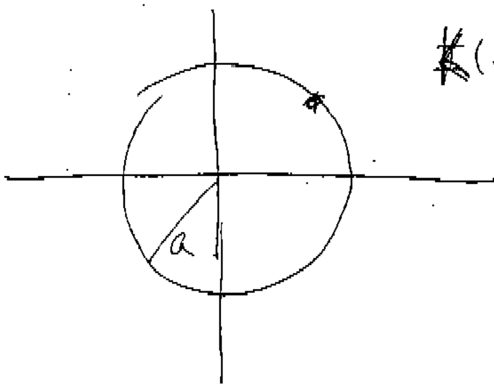
$$(\vec{\nabla} \times \vec{A})_\phi = \frac{\mu_0 I}{2\pi s}$$

But $(\vec{\nabla} \times \vec{A})_\phi = -\frac{c}{s}$

$$\Rightarrow \quad c = -\frac{\mu_0 I}{2\pi}$$

$$\Rightarrow \quad \boxed{A_z = -\frac{\mu_0 I}{2\pi} \ln s}$$

Example 2 :



$\vec{K}(\phi) = K_0 \cos\phi \; \hat{z}$

Find $\vec{A}$ & $\vec{B}$ Inside circle

$(s < a)$

For $s < a$ $\qquad \nabla^2 A_z = 0$

$$\Rightarrow A_z = C_1 \ln s + \sum_{n=1}^{\infty} (A_n s^n + B_n s^{-n})(D_n \cos n\phi + E_n \sin n\phi)$$

For A_z to be well behaved at $s = 0$

$$\Rightarrow \quad C_1 = B_n = 0$$

$$\Rightarrow A_z = \sum_{n=1}^{\infty} A_n s^n (D_n \cos n\phi + E_n \sin n\phi)$$

Note that this form satisfies $\vec{\nabla} \cdot \vec{A} = \dfrac{\partial A_z}{\partial z} = 0$

since A_z is not a function of z

4/16/04

Ampere's law:

$$\oint B_\phi \cdot dl = \mu_0 I = \mu_0 K_0 \cos\phi \cdot a\, d\phi$$

$$2B_\phi \cdot a\, d\phi = \mu_0 K_0 \cos\phi \cdot a\, d\phi$$

since B_ϕ reverses sign outside

$$B_\phi = -\frac{\mu_0 K_0 \cos\phi}{2}$$

Now $B_\phi = (\nabla \times \vec{A})_\phi = \left(\frac{\partial A_s}{\partial z} - \frac{\partial A_z}{\partial s} \right) = -\frac{\partial A_z}{\partial s}$

$$\Rightarrow -\frac{\mu_0 K_0 \cos\phi}{2} = -\sum_{n=1}^{\infty} n A_n s^{n-1} (D_n \cos n\phi + E_n \sin n\phi)$$

$\Rightarrow$ $E_n = 0$,
$D_n = 0 \quad \forall n \neq 1$
$A_1 D_1 = \frac{\mu_0 K_0}{2}$

$$\Rightarrow A_z = \frac{\mu_0 K_0}{2} s \cos\phi = \frac{\mu_0 K_0}{2} x$$

$$\Rightarrow \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & \frac{\mu_0 K_0}{2} x \end{vmatrix} = -\frac{\mu_0 K_0}{2} \hat{y}$$

uniform!

Example 3: Long Solenoid $\quad B_z = \mu_0 n I$ inside

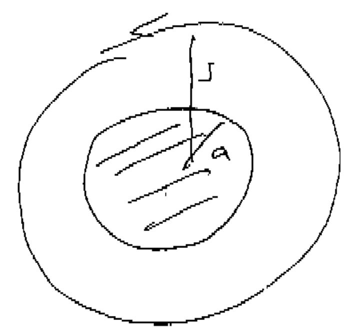
$$\oint \vec{A}\cdot d\vec{l} = \int \vec{B}\cdot da = (\mu_0 n I)(\pi s^2)$$

$$A_\phi = \frac{\mu_0 n I s}{2} \qquad s < a$$

Outside the solenoid, $B_z = 0$

but $(2\pi s) A_\phi = \pi a^2 \cdot \mu_0 n I$

$$A_\phi = \frac{\mu_0 n I a^2}{2s} \qquad s > a$$



Therefore $A_\phi \neq 0$ even though

$$B_z = (\vec{\nabla}\times\vec{A})_z = \frac{1}{s}\frac{\partial}{\partial s}(sA_\phi) = 0$$

The Lagrangian / Hamiltonian formulation of Mechanics involves the potential $\vec{A}$, V rather than the fields $\vec{E}$, $\vec{B}$

$$H = \frac{p^2}{2m} + eV \rightarrow \frac{1}{2m}(\vec{p} - e\vec{A})^2 + eV$$

Aharonov - Bohm effect.

4/19/04

Boundary Conditions for $\vec{B}$

Recall that we are solving $\nabla^2 \vec{A} = -\mu_0 \vec{J}$

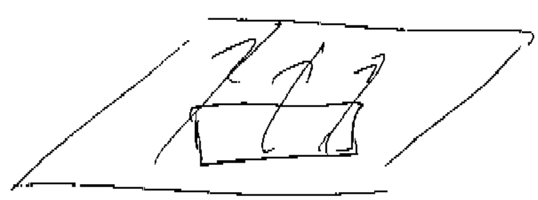
(analogous to $\nabla^2 V = -\rho/\epsilon_0$)

Soln is subject to b.c. (cf. E-field)

- $\vec{\nabla} \cdot \vec{B} = 0 \quad \Rightarrow \quad \boxed{B^{\perp}_{above} = B^{\perp}_{below}}$

- $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \quad \Rightarrow \quad \oint \vec{B} \cdot d\vec{l} = \mu_0 I$

$$\Rightarrow \quad B^{\parallel}_{above} - B^{\parallel}_{below} = \mu_0 K$$



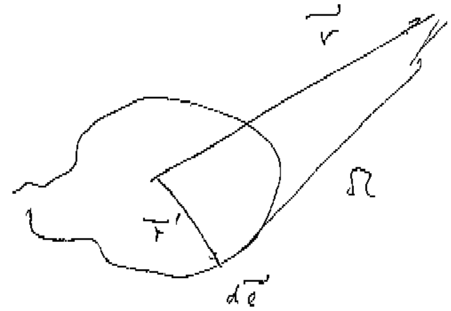
$$\Rightarrow \quad \boxed{B^{\parallel}_{above} - B^{\perp}_{below} = \mu_0 K}$$

- $\vec{A}$ is continuous

$$\boxed{\vec{A}_{above} = \vec{A}_{below}}$$

Multipole Expansion

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi} \oint \frac{d\vec{l}\,'}{\Omega}$$



$$= \frac{\mu_0 I}{4\pi} \oint \frac{d\vec{l}\,'}{|\vec{r}-\vec{r}\,'|}$$

Consider $r > r'$

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{(r^2+r'^2-2rr'\cos\theta)^{1/2}} = \frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l P_l(\cos\theta')$$

So $\vec{A}(\vec{r}) = \dfrac{\mu_0 I}{4\pi r} \displaystyle\oint \sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l P_l(\cos\theta)\, d\vec{l}\,'$

$$= \frac{\mu_0 I}{4\pi r}\left\{ \oint d\vec{l}\,' + \frac{1}{r}\oint r'\cos\theta'\, d\vec{l}\,' + \frac{1}{r^2}\oint r'^2 P_2(\cos\theta)\, d\vec{l}\,' + \cdots \right\}$$

Obviously $\oint d\vec{l}\,' = 0$ $\Rightarrow$ no monopole term
(remember the starting pt of $\vec{B} = \vec{\nabla}\times\vec{A}$ is $\vec{\nabla}\cdot\vec{B} = 0$)

The leading term is the dipole term

$$\vec{A}_{dipole}(\vec{r}) = \frac{\mu_0 I}{4\pi r^2} \oint r' \cos\theta' \, d\vec{l}'$$

$$= \frac{\mu_0 I}{4\pi r^2} \oint \vec{r}' \cdot \hat{r} \, d\vec{l}'$$

We can show that

$$\oint (\hat{r} \cdot \vec{r}') \, d\vec{l}' = -\hat{r} \times \int d\vec{a}' \qquad [\text{Ch } 1]$$

Aside: This follows from a variation of Stokes's thm. Let $\vec{v} = \vec{c}\,T$ in Stokes's thm

$$\int_S (\vec{\nabla} T) \times d\vec{a} = -\oint T \, d\vec{l} \qquad \vec{c} = \text{constant vector.}$$

Now Let $T = \vec{c} \cdot \vec{r}$ where $\vec{c}$ = constant

$$\oint (\vec{c} \cdot \vec{r}) \, d\vec{l} = \vec{a} \times \vec{c} \quad \text{where}$$

$$\vec{a} = \frac{1}{2} \oint \vec{r} \times d\vec{l}$$

Therefore $\vec{A}_{dipole}(\vec{r}) = \frac{\mu_0 I}{4\pi r^2}\left(\int d\vec{a}\right)\times\hat{r}$

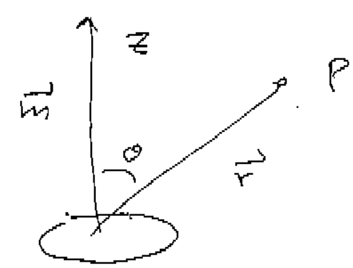
Define the magnetic dipole moment $\vec{m} = I\int d\vec{a} = I\vec{a}$

and $\vec{A}_{dipole}(\vec{r}) = \frac{\mu_0 I}{4\pi r^2}(\vec{m}\times\hat{r})$

Compare to $V_{dipole} = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}$ $\quad \vec{p} \Leftrightarrow \vec{m}$

dot $\Leftrightarrow$ cross

Let $\vec{m}$ be along z



then $\vec{m}\times\vec{r} = m\sin\theta\,\hat{\phi}$

$$\vec{A}_{dipole}(\vec{r}) = \frac{\mu_0}{4\pi r^2} m\sin\theta\,\hat{\phi}$$

$$\vec{B} = \vec{\nabla}\times\vec{A} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\sin\theta A_\phi\,\hat{r} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\,\hat{\theta}$$

$$= \frac{\mu_0 m}{4\pi}\left\{\frac{\hat{r}}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\sin^2\theta}{r^2}\right) - \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\sin\theta}{r^2}\right)\hat{\theta}\right\}$$

$$= \frac{\mu_0 m}{4\pi}\left(\frac{2\cos\theta}{r^3}\hat{r} + \frac{\sin\theta}{r^3}\hat{\theta}\right)$$

$$= \frac{\mu_0 m}{4\pi r^3}(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}) \quad \text{(Eq 5.86)}$$

More compactly, $\vec{m} = m\hat{z} = m(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta})$

$$\vec{B}_{dipole}(\vec{r}) \simeq \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}\right]$$

Eq. 5.87

Compare

$$\vec{E}_{dipole}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}\right]$$

Eq. 3.104

They have the same form!