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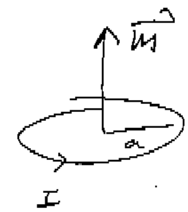
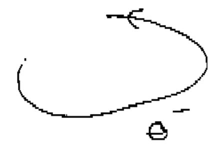
Chapter 6 Magnetic Fields in Matter

Magnetic moment $\vec{m} = I\vec{A}$



Magnetic moment of an electron:

$$I = -\frac{e}{\tau}$$



where τ = period of orbit

$$= \frac{2\pi a}{v}$$ a = orbit radius v = electron vel.

$$\Rightarrow I = \frac{ev}{2\pi a} = \frac{e\,l}{2\pi a^2 m_e}$$

$l = m_e v a$ = angular momentum

l = angular momentum is measured in units of $\hbar$, the Planck constant divided by 2π

$$m = IA = I\pi a^2 = \frac{e\hbar L}{2m_e}$$ $L = 0, 1, 2, \ldots$ integers

The combination $\frac{e\hbar}{2m_e}$ is called Bohr magneton

In general (for things other than electrons, say)

$$m = g\left(\frac{e\hbar}{2m_e}\right) l$$ g = Lande g factor

The magnetic moment divided by c

$$\frac{m}{c} = g \frac{e\hbar}{2m_e c} \mathbf{s}$$

has the same dimensions as an electric dipole moment.

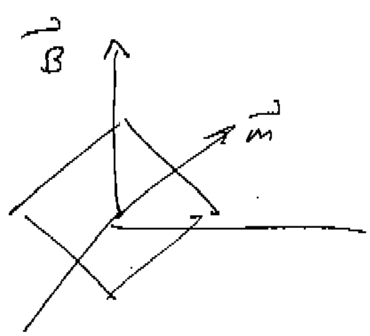
$$\frac{\hbar}{m_e c} = \lambda\!\!\!^{-}_c = \frac{\lambda_c}{2\pi} = \text{Compton wavelength}$$

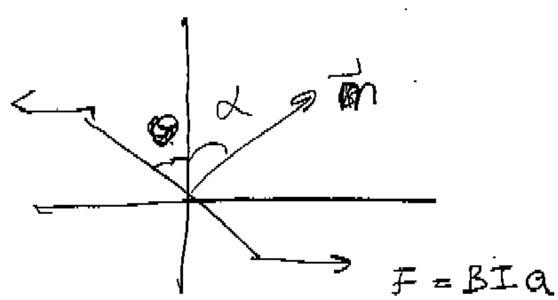
$$\left[\frac{\hbar}{m_e c}\right] = \text{distance}$$

Therefore

$\frac{m}{c}$ & p have the same dimensions.

Magnetic moments tend to align with $\vec{B}$





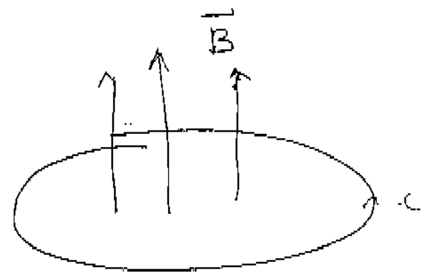
Torque

$$N = aF\cos\theta = aF\sin\alpha$$
$$= B(Ia^2)\sin\alpha$$
$$= \vec{m} \times \vec{B}$$

Effect of Magnetic Field on Atomic Orbits

For $B=0$, $\qquad \dfrac{e^2}{4\pi\epsilon_0 R^2} = m\dfrac{v^2}{R}$

For $B \neq 0$



$$\frac{e^2}{4\pi\epsilon_0 R^2} + e\bar{v}B = m\frac{\bar{v}^2}{R} \qquad \bar{v} = \text{new speed}$$

The new speed $\bar{v} > v$:

$$e\bar{v}B = \frac{m}{R}(\bar{v}^2 - v^2) = \frac{m}{R}(\bar{v}+v)(\bar{v}-v)$$

Assume $\Delta v = \bar{v} - v$ is small.

$$\Delta v = \frac{eRB}{2m}$$

$$\Delta m = \Delta I \cdot A = \frac{-e}{2\pi R} \cdot \pi R^2 \cdot \Delta v = \frac{e(\Delta v)R}{2}$$

$$= \frac{-e^2R^2}{4m}\vec{B} \qquad \text{opposite to } \vec{B} \Rightarrow \text{diamagnetism}$$

Some definitions:

Paramagnetism is simply the alignment of electron's magnetic moment with external field.

Usually Diamagnetism is quite small, and usually observed in atoms with an even number of electrons, where there is no paramagnetism because of Pauli Exclusion Principle

Ferromagnetism is a particularly strong version of paramagnetism. Magnetization ~~[illegible]~~ is retained even when the external field is removed.

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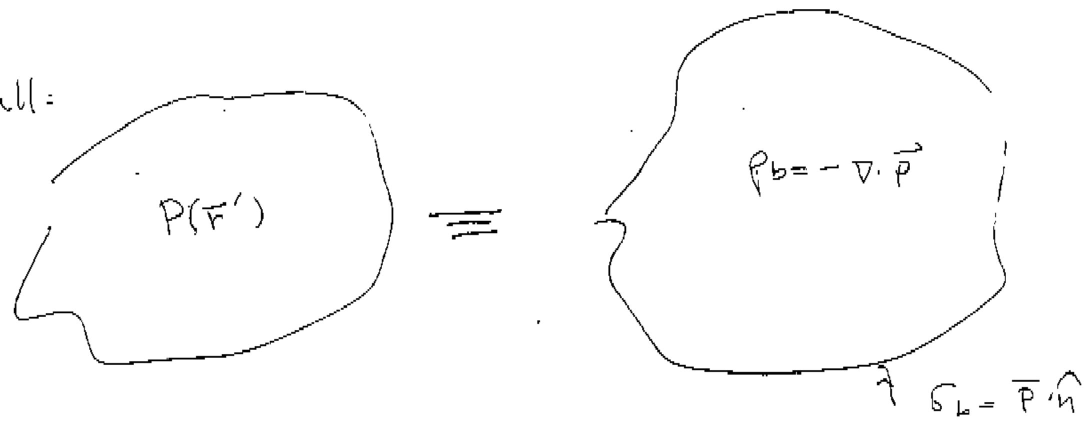
Bound Currents

Recall:



Now, carry out similar analysis for $\vec{M}$

For a single magnetic moment $\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{r}}{r^2}$

So for a volume distribution

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{M} \times \hat{r}}{r^2} d\tau' \qquad \vec{r} = \vec{r} - \vec{r}' \text{ as usual}$$

We use argument similar to the electrical case:

$$\nabla' \frac{1}{r} = \frac{\hat{r}}{r^2} \qquad \text{which we showed earlier}$$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \vec{M} \times \vec{\nabla}'\left(\frac{1}{r}\right) d\tau'$$

The idea is to convert this to $\frac{1}{r}$ potential

Now $\vec{\nabla} \times (\vec{M} V) = V \vec{\nabla} \times \vec{M} - \vec{M} \times \vec{\nabla} V$

So $\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{\nabla}' \times M}{r} d\tau' - \frac{\mu_0}{4\pi} \int \vec{\nabla}' \times \left(\frac{\vec{M}}{r}\right) d\tau'$

$$= \frac{\mu_0}{4\pi} \int \frac{\vec{\nabla}' \times M}{r} d\tau' + \frac{\mu_0}{4\pi} \int \frac{\vec{M} \times d\vec{a}}{r}$$

Since $\int_{Vol} \vec{\nabla} \times \vec{A} \, d\tau = - \int_{surface} \vec{A} \times d\vec{a}$

Define $\vec{J}_b = \vec{\nabla} \times M$ = induced volume current

$\vec{K}_b = \vec{M} \times \hat{n}$ = induced surface current

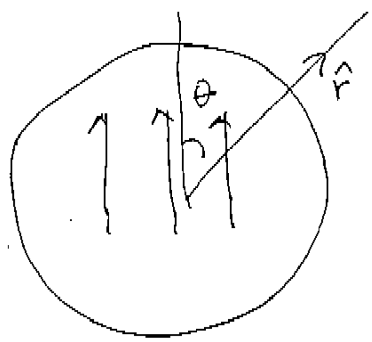
So $\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}_b(\vec{r}') d\tau'}{r} + \frac{\mu_0}{4\pi} \int \frac{\vec{K}_b da'}{r}$

Compare with

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho_b}{r} d\tau' + \frac{1}{4\pi\epsilon_0} \int \frac{\sigma_b}{r} da'$$

$\rho_b = -\vec{\nabla} \cdot \vec{P} \qquad \sigma_b = \vec{P} \cdot \hat{n}$

<u>Example</u> : Uniformly magnetized sphere $\vec{M} = M\hat{z}$



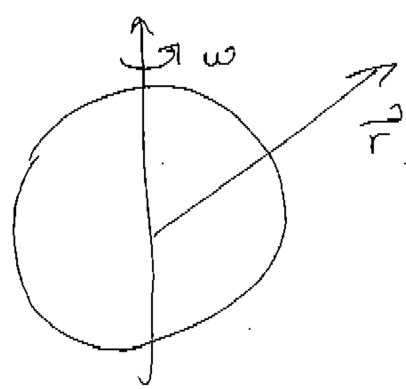
Find $A(\vec{r})$

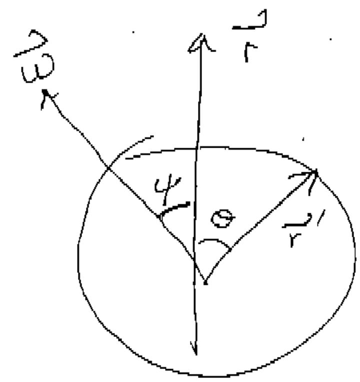
Soln: $\vec{\nabla} \times \vec{M} = 0 \Rightarrow \vec{J}_b = 0$

$$\vec{M} \times \hat{r} = M \sin\theta\, \hat{\phi} \Rightarrow \vec{K}_b = M \sin\theta\, \hat{\phi}$$

same as a spinning spherical shell

$$\vec{K} = \sigma \vec{v} = \sigma \vec{\omega} \times \vec{r}\,'$$

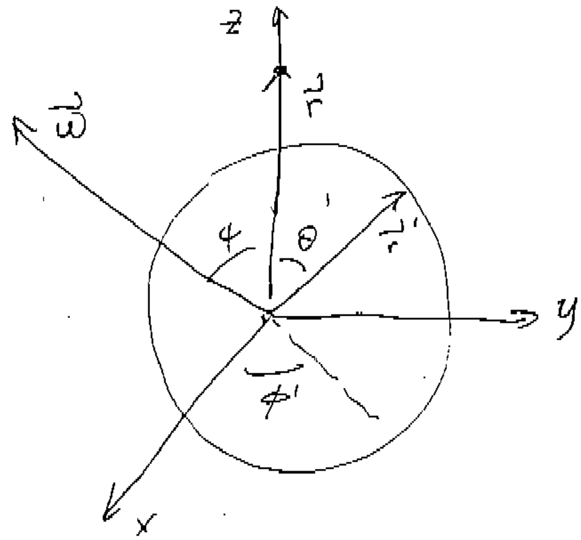




Turns out is easier to align the vector $\vec{r}$ along the z-axis.

$\vec{r}$ = position of observer.

Hence $\vec{\omega} = \omega\,(\cos\psi\,\hat{z} + \sin\psi\,\hat{z})$



$$\vec{K} = \sigma\,\vec{\omega}\times\vec{r}\,' = \sigma \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \omega\sin\psi & 0 & \omega\cos\psi \\ R\sin\theta'\cos\phi' & R\sin\theta'\sin\phi' & R\cos\theta' \end{vmatrix}$$

$$= \sigma R\omega \Big[-(\cos\psi\sin\theta'\sin\phi')\,\hat{x} + (\cos\psi\sin\theta'\cos\phi' - \sin\psi\cos\theta')\,\hat{y} + (\sin\psi\sin\theta'\sin\phi')\,\hat{z} \Big]$$

Remember unprimed variables refer to observation point
primed " . . . . sources.

A lot of terms drop out in $A(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{K\,da'}{r}$

This is because

$$\int_0^{2\pi} \sin\phi' \, d\phi' = \int_0^{2\pi} \cos\phi' \, d\phi' = 0$$

The only surviving term is

$$A(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\sigma \, \vec{\omega} \times \vec{r}\,' \, da'}{|\vec{r} - \vec{r}\,'|} \qquad |\vec{r} - \vec{r}\,'|^2 = R^2 + r^2 - 2Rr\cos\theta'$$

$$= \frac{\mu_0}{4\pi} \, \sigma R \omega \int \frac{-\sin\psi \cos\theta' \cdot R^2 \sin\theta' \, d\theta' \, d\phi'}{(R^2 + r^2 - 2Rr\cos\theta')^{1/2}}$$

$$= -\frac{\mu_0}{4\pi} (\sigma R \omega) \sin\psi \cdot R^2 \cdot 2\pi \underbrace{\int \frac{\sin\theta' \cos\theta' \, d\theta'}{(R^2 + r^2 - 2Rr\cos\theta')^{1/2}}}_{\text{evaluate this integral}}$$

Let $u = \cos\theta'$

$$\int_{-1}^{1} \frac{u \, du}{(R^2 + r^2 - 2Rru)^{1/2}} = \text{integral to be evaluated,}$$

We know how to integrate

$$\int_{-1}^{1} \frac{du}{(R^2 + r^2 - 2Rru)^{1/2}} = -\frac{1}{rR} (r^2 + R^2 - 2Rru)^{1/2} \Big|_{-1}^{1}$$

Using integration by parts

$$\int_{-1}^{1} u\,dv = uv\Big|_{-1}^{1} - \int_{-1}^{1} v\,du$$

We can calculate

$$\int_{-1}^{1} \frac{u\,du}{(R^2+r^2-2rRu)^{1/2}} = \int_{-1}^{1} u \; dv$$

where $dv = \dfrac{du}{(R^2+r^2-2Rru)^{1/2}} \Rightarrow v = -\dfrac{1}{rR}(r^2+R^2-2rRu)$

Therefore

$$\int_{-1}^{1} \frac{u\,du}{(R^2+r^2-2rRu)^{1/2}} = -\frac{u}{rR}(r^2+R^2-2rRu)^{1/2}\Big|_{-1}^{1}$$

$$+\frac{1}{rR}\int_{-1}^{1}(r^2+R^2-2rRu)^{1/2}du$$

$$= \left[-\frac{u}{rR}(r^2+R^2-2rRu)^{1/2} - \frac{1}{\cancel{2}r^2R^2}\times\frac{\cancel{2}}{3}(r^2+R^2-2rRu)^{3/2}\right]_{-1}^{1}$$

$$= \frac{(r^2+R^2-2rRu)^{1/2}}{3R^2r^2}\Big[-3rRu - r^2-R^2+2rRu\Big]_{-1}^{1}$$

$$= -\frac{(r^2+R^2-2Rru)^{1/2}}{3R^2r^2}\,(R^2+r^2+rRu)\Bigg|_{-1}^{1}$$

$$= -\frac{1}{3R^2r^2}\Big[(R^2+r^2+Rr)\,|R-r| - (R^2+r^2-Rr)(R+r)\Big]$$

$$= \begin{cases} \dfrac{3r}{3R^2} & R>r \quad \text{inside} \\[2ex] \dfrac{2R}{3r^2} & R<r \quad \text{outside} \end{cases}$$

Since $\vec{\omega}\times\vec{r} = -\omega r \sin\psi\,\hat{y}$

$$\Rightarrow \quad \vec{A}(\vec{r}) = \begin{cases} \dfrac{\mu_0 R\sigma}{3}\,(\vec{\omega}\times\vec{r}) & \text{inside} \\[2ex] \dfrac{\mu_0 R^4\sigma}{3r^3}\,(\vec{\omega}\times\vec{r}) & \text{outside} \end{cases}$$

Match with the physical quantity $\vec{M}$

$$\vec{K} = \sigma\vec{v} = \sigma\,\vec{\omega}\times\vec{r}\,' = \sigma\omega R\sin\theta\,\hat{\phi}$$

$$\vec{K} = M\sin\theta\,\hat{\phi}$$

$\Rightarrow$ Identify $\quad \vec{M} \equiv R\sigma\vec{\omega}$