PHY 322 Fall 2004 Midterm II Solns

Problem 1. In rectangular coordinates, the Laplace Eqn is

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0$$

Assume $U(x,y) = X(x)\,Y(y)$, we have

$$Y \frac{d^2 X}{dx^2} + X \frac{d^2 Y}{dy^2} = 0$$

i.e. $$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2}$$

$\because$ $Y(0) = Y(b) = 0$, Y is sinusoidal

$\therefore$ Let $\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = m^2$

$\therefore$ For Y, we have $Y = C_m \cos my + D_m \sin my$

For X, $X = A_m e^{mx} + B_m e^{-mx}$

Thus $$U(x,y) = \sum_{m=0}^{\infty} \left(A_m e^{mx} + B_m e^{-mx}\right)\left(C_m \cos my + D_m \sin my\right)$$

Given boundary conditions:

① $U(x,0)=0$

$$\therefore \sum_{m=0}^{\infty}\left(A_m e^{mx}+B_m e^{-mx}\right)C_m = 0$$

$$\therefore \quad C_m=0$$

② $U(x,b)=0$

$$\therefore \sum_{m=0}^{\infty}\left(A_m e^{mx}+B_m e^{-mx}\right)\left(D_m \sin mb\right)=0 \qquad (\because C_m=0)$$

$$\therefore \quad mb=n\pi \qquad n=0,\pm 1,\pm 2 \cdots$$

$$m=\frac{n\pi}{b} \quad ✓$$

Hence $U(x,y)=\sum_{n=1}^{\infty}\left(a_n e^{\frac{n\pi}{b}x}+b_n e^{-\frac{n\pi}{b}x}\right)\sin\frac{n\pi}{b}y$

③ At $x=0$. $\frac{\partial U}{\partial x}=0$

$$\therefore \sum_{n=1}^{\infty}\frac{n\pi}{b}\left(a_n-b_n\right)\sin\frac{n\pi}{b}y=0$$

✓

$$\therefore \quad a_n=b_n$$

Hence $U(x,y)=\sum_{n=1}^{\infty}\left(e^{\frac{n\pi x}{b}}+e^{-\frac{n\pi x}{b}}\right)a_n\sin\frac{n\pi}{b}y$

④ when $x=a$ $U=U_0$ ✓

$$\therefore \quad U_0=\sum_{n=1}^{\infty}\left(e^{\frac{n\pi a}{b}}+e^{-\frac{n\pi a}{b}}\right)a_n\sin\frac{n\pi y}{b}$$

$$\because \quad U(a,0) = U(a,b)$$

$$\left(e^{\frac{n\pi a}{b}} + e^{\frac{-n\pi a}{b}}\right) a_n = \frac{2}{b}\int_0^b U_0 \sin \frac{n\pi}{b} y \, dy$$

$$= \frac{2U_0}{b}\,\frac{b}{n\pi}\, -\cos \frac{n\pi}{b} \Big|_0^b$$

$$= \frac{2U_0}{n\pi}\left(1 - \cos n\pi\right)$$

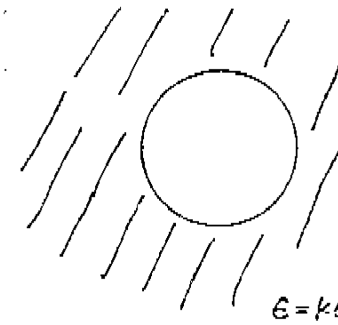
$$= \begin{cases} \dfrac{4U_0}{n\pi} & \text{if } n \text{ is odd} \\ 0 & \text{" " " even} \end{cases}$$

$$\therefore \quad a_n = \begin{cases} \dfrac{2V_0}{n\pi \cosh \frac{n\pi a}{b}} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

Hence $$U = \frac{4U_0}{\pi} \sum_{n\ \text{odd}} \frac{1}{n} \frac{\cosh \frac{n\pi x}{b}}{\cosh \frac{n\pi a}{b}} \sin \frac{n\pi y}{b}$$ #

Problem 2.

$$U_1 = (A_1 r + B_1/r^2)\cos\theta \qquad r > R$$
$$U_2 = (A_2 r + B_2/r^2)\cos\theta \qquad r < R$$



$\epsilon = k\epsilon_0$

$\vec{E}_{in} = \frac{3k}{2k+1}\vec{E}$

since $k > 1$

E_{in} in the hole $> E$

why ?

(i) U_2 finite at $r=0 \Rightarrow B_2 = 0$

(ii) $r \to \infty \quad U_1 \to -Ez = -Er\cos\theta$

$$\Rightarrow A_1 = -E$$

(iii) $U_1|_{r=R} = U_2|_{r=R}$ (U continuous)

$$\Rightarrow \quad -ER + \frac{B_1}{R^2} = A_2 R \qquad (1)$$

(iv) $\epsilon \frac{\partial U_1}{\partial r}\Big|_{r=R} = \epsilon_0 \frac{\partial U_2}{\partial r}\Big|_{r=R}$ ($D_\perp$ continuous)

$$\Rightarrow \quad K\left(-E - \frac{2B_1}{R^3}\right) = A_2 \qquad (2)$$

Sub (2) into (1),

$$-ER + \frac{B_1}{R^2} = KR\left(-E - \frac{2B_1}{R^3}\right)$$
$$(1+2K)\frac{B_1}{R^2} = (1-K)RE$$
$$\therefore B_1 = -\frac{K-1}{2K+1} E R^3$$

From (1)

$$A_2 = -E + \frac{1}{R^3}\left[-\frac{K-1}{2K+1} E R^3\right]$$
$$= -E\left[1 + \frac{K-1}{2K+1}\right] = -\frac{3KE}{2K+1}$$

$$\therefore \quad U(r,\theta) = -E\left(r + \frac{K-1}{2K+1}\frac{R^3}{r^2}\right)\cos\theta \qquad \text{for } r > R$$
$$= -\frac{3KE}{2K+1} r\cos\theta . \qquad \text{for } r \le R$$

$$\vec{E}_{in} = -\nabla U_2 = \frac{3KE}{2K+1}\hat{e}_z \frac{\partial}{\partial z} z = \frac{3K\vec{E}}{2K+1} \quad \#$$