PHY 322 Midterm II Solutions

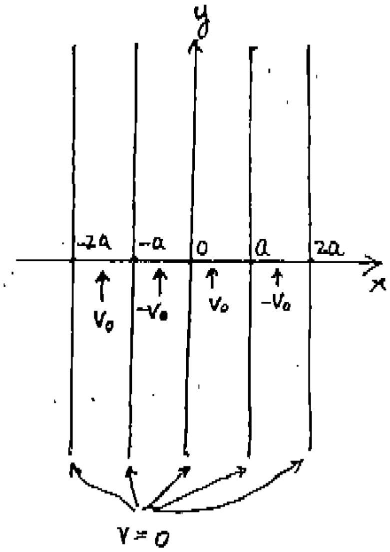
1. Boundary condition is periodic in x

$V(x=na, y) = 0 \quad n = 0, 1, 2, \ldots$

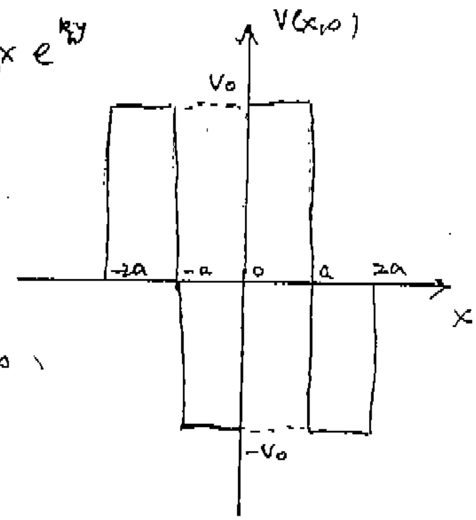
$\Rightarrow$ the solution is oscillatory in x.

$V(0, y)$ is an odd function in x

$\Rightarrow$ the solution can be expressed in Fourier sine series.



$$\therefore V(x,y) = \sum_{n=1}^{\infty} a_n \sin k_n x \, e^{-k_n y} + b_n \sin k_n x \, e^{k_n y}$$



One <u>implicit</u> boundary condition is

<u>$V(x, y \to \pm\infty) = 0$</u>. This is true

<u>otherwise</u> the exponential dependence implies,

$V(x, y \to \pm\infty) \to \infty$ for $x \neq na$

$V(x, y \to \pm\infty) = 0$ for $x = na$

This implies an infinite E field for large y. For this reason $\cosh ky$, $\sinh ky$ cannot satisfy the boundary condition.

$$\therefore V(x,y) = \sum_{n=1}^{\infty} a_n \sin k_n x \, e^{-k_n |y|} \quad \text{(The solution is sym in } \pm y\text{)}$$

$V(x=na, y) = 0 \quad$ for $\quad k_n = \frac{n\pi}{a}$

$$V(x,0) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{a} x$$

②

Multiply both side by $\sin\frac{m\pi x}{a}$ and integrate over 1 period ($V(x,0)$ has a period of $2a$)

$$\int_{-a}^{a} V(x,0)\sin\frac{m\pi x}{a}\,dx = \sum_{n=1}^{\infty} a_n \int_{-a}^{a} \sin\frac{n\pi x}{a}\sin\frac{m\pi x}{a}\,dx$$

$$= \sum_{n=1}^{\infty} a_n\,\delta_{mn}\,a = a_m a$$

$$\int_{-a}^{a} V(x,0)\sin\frac{m\pi x}{a}\,dx = 2\int_{0}^{a} V_0 \sin\frac{m\pi x}{a}\,dx \qquad (V(x,0) \text{ is odd fcn})$$

$$= \frac{2aV_0}{m\pi}(1-\cos m\pi) = \begin{cases} 0, & m \text{ is even} \\ \frac{4V_0 a}{m\pi}, & m \text{ is odd} \end{cases}$$

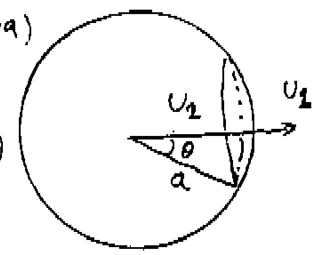
$$\therefore\quad V(x,y) = \sum_{\substack{n=1 \\ \text{odd } n}}^{\infty} \frac{4V_0}{n\pi}\sin\frac{n\pi x}{a}\, e^{-\frac{n\pi|y|}{a}}$$

#

2

$$U_1 = \sum_{l=0}^{\infty} \left[A_l r^l + B_l r^{-(l+1)} \right] P_l(\cos\theta) \quad (r>a)$$

$$U_2 = \sum_{l=0}^{\infty} \left[A_l' r^l + B_l' r^{-(l+1)} \right] P_l(\cos\theta) \quad (r<a)$$



(not a conductor!)

Boundary conditions :

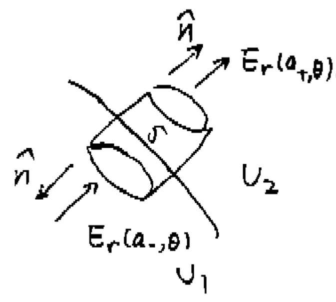
(i) $U_1(r \to \infty, \theta) \to 0$

(ii) $U_2(r \to 0, \theta)$ is finite.

(iii) $U_1(a, \theta) = U_2(a, \theta)$

(iv) $E_r(a_+, \theta) - E_r(a_-, \theta) = \frac{1}{\epsilon_0} \underbrace{\cos 2\theta}_{}$ — σ free surface charge

E_r ↑ normal component.



* By (i), $A_l = 0$ forall l

$$\therefore U_1 = \sum_{l=0}^{\infty} B_l r^{-(l+1)} P_l(\cos\theta)$$

* By (ii), $B_l' = 0$ for all l

$$\therefore U_2 = \sum_{l=0}^{\infty} A_l' r^l P_l(\cos\theta)$$

* By (iii) comparing the coefficient of $P_l(\cos\theta)$

$$B_l a^{-(l+1)} = A_l' a^l \Rightarrow A_l' = B_l a^{-(2l+1)}$$

or $$U_2 = \sum_{l=0}^{\infty} \frac{B_l}{a^{2l+1}} r^l P_l(\cos\theta)$$

$$E_r(a+,\theta) = -\frac{\partial U_1}{\partial r}\Big|_a = \sum_{\ell=0}^{\infty} \frac{(\ell+1)B_\ell}{a^{\ell+2}} P_\ell(\cos\theta)$$

$$E_r(a-,\theta) = -\frac{\partial U_2}{\partial r}\Big|_a = \sum_{\ell=0}^{\infty} -\frac{\ell B_\ell}{a^{2\ell+1}} \cdot a^{\ell-1} P_\ell(\cos\theta)$$

$$\cos 2\theta = 2\cos^2\theta - 1 = -\frac{1}{3} + \frac{4}{3} P_2(\cos\theta) \qquad \left(\begin{array}{l} P_0(\cos\theta) = 1 \\ P_2(\cos\theta) \\ = \frac{1}{2}(3\cos^2\theta - 1) \end{array} \right)$$

∴ By (iv) $\quad E_r(a+,\theta) - E_r(a-,\theta)$

$$= \sum_{\ell=0}^{\infty} \frac{(2\ell+1)B_\ell}{a^{\ell+2}} P_\ell(\cos\theta) = \frac{1}{\varepsilon_0}\left(-\frac{1}{3}P_0(\cos\theta) + \frac{4}{3}P_2(\cos\theta)\right)$$

$$\Rightarrow \begin{cases} \dfrac{B_0}{a^2} = -\dfrac{1}{3\varepsilon_0} \\ \dfrac{5B_2}{a^4} = \dfrac{4}{3\varepsilon_0} \end{cases}, \quad B_\ell = 0 \text{ otherwise.}$$

$$\therefore \quad B_0 = -\frac{a^2}{3\varepsilon_0}, \quad B_2 = \frac{4a^4}{15\varepsilon_0}, \quad B_\ell = 0, \quad \ell \neq 0, 2$$

$$\therefore U_1 = \frac{1}{\varepsilon_0}\left\{ -\frac{a^2}{3r} + \frac{4}{15}\frac{a^4}{r^3} P_2(\cos\theta) \right\}$$

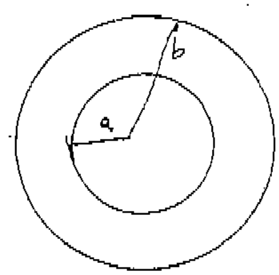
$$= \frac{1}{\varepsilon_0}\left\{ -\frac{a^2}{3r} + \frac{2}{15}\frac{a^4}{r^3}(3\cos^2\theta - 1) \right\}$$

#

\# 3.

$$4\pi\epsilon_0 U(r,\theta) = q\ln r + \sum_{n=0}^{\infty} (A_n r^n + B_n r^{-n})\sin n\theta + \sum_{n=0}^{\infty} (C_n r^n + D_n r^{-n})\cos n\theta$$

Boundary conditions :
$$\begin{cases} U(a,\theta) = V_a \sin\theta \\ U(b,\theta) = V_b \cos 3\theta \end{cases}$$



The boundary conditions of U involves $\sin\theta$, $\cos 3\theta$ only. The <u>linear independence</u> of the set $\{\sin n\theta, \cos n\theta\}_{n=0}^{\infty}$ implies $U(r,\theta)$ can be written as

$$4\pi\epsilon_0 U(r,\theta) = \left(A_1 r + \frac{B_1}{r}\right)\sin\theta + \left(C_3 r^3 + \frac{D_3}{r^3}\right)\cos 3\theta$$

At $r=a$
$$\frac{1}{4\pi\epsilon_0}\left[\left(A_1 a + \frac{B_1}{a}\right)\sin\theta + \underbrace{\left(C_3 a^3 + \frac{D_3}{a^3}\right)}_{=0}\cos 3\theta\right] = V_a \sin\theta$$

At $r=b$
$$\frac{1}{4\pi\epsilon_0}\left[\underbrace{\left(A_1 b + \frac{B_1}{b}\right)}_{=0}\sin\theta + \left(C_3 b^3 + \frac{D_3}{b^3}\right)\cos 3\theta\right] = V_b \cos 3\theta$$

$$\therefore \begin{cases} A_1 a + \dfrac{B_1}{a} = 4\pi\epsilon_0 V_a, & C_3 a^3 + \dfrac{D_3}{a^3} = 0 \\ A_1 b + \dfrac{B_1}{b} = 0, & C_3 b^3 + \dfrac{D_3}{b^3} = 4\pi\epsilon_0 V_b \end{cases}$$

$$\Rightarrow \quad A_1 = \frac{4\pi\epsilon_0 V_a}{(a - b^2/a)}, \quad B_1 = \frac{-4\pi\epsilon_0 V_a b^2}{(a - b^2/a)}$$

$$C_3 = \frac{4\pi\epsilon_0 V_b}{b^3 - a^6/b^3}, \quad D_3 = \frac{-4\pi\epsilon_0 V_b a^6}{b^3 - a^6/b^3}$$