P322 Midterm I Solutions

Problem 1

(a) By Gauss's law $\oint \vec{E} \cdot d\vec{a} = \dfrac{Q_{enclosed}}{\epsilon_0}$

$$\Rightarrow \quad E \cdot 4\pi r^2 = \frac{Q_{enclosed}}{\epsilon_0}$$

For $0 < r < R$ $\quad Q_{enclosed} = \int_0^r \rho \left(4\pi r^2 dr\right)$

$$= \int_0^r 4\pi \frac{A}{r} r^2 dr = 4\pi A \frac{r^2}{2}$$

$$= 2\pi A r^2$$

$$\Rightarrow \quad \boxed{\vec{E} = \frac{A}{2\varepsilon_0} \hat{r}}$$

$r > R$ $\quad Q_{enclosed} = 2\pi A R^2$

$$\Rightarrow \quad \boxed{\vec{E} = \frac{AR^2}{2\varepsilon_0 r^2} \hat{r}}$$

(b) Potential ;

For $r > R$ $V(r) = -\int_{\infty}^{r} \vec{E} \cdot d\vec{r}$

$$= -\int_{-\infty}^{r} \frac{A}{2\varepsilon_0} \frac{R^2}{r^2} dr$$

$$= -\frac{A}{2\varepsilon_0} \left(-\frac{R^3}{r} \right) \Big|_{\infty}^{r}$$

$$= \frac{AR^2}{2\varepsilon_0 r}$$

For $r < R$ $V(r) = -\int_{\infty}^{r} \vec{E} \cdot d\vec{r}$

$$= -\int_{\infty}^{R} E \, dr - \int_{R}^{r} E(r) dr$$

$$= \frac{AR}{2\varepsilon_0} - \int_{R}^{r} \frac{A}{2\varepsilon_0} dr$$

$$= \frac{A}{2\varepsilon_0} \left[R - (r - R) \right]$$

$$= \frac{A}{2\varepsilon_0} (2R - r)$$

(c) $W = \int \frac{1}{2} \varepsilon_0 E^2 \, d\tau$

$= \frac{1}{2} \varepsilon_0 \int E^2 \cdot 4\pi r^2 dr$

$= 2\pi \varepsilon_0 \left[\int_0^R \frac{A^2}{4\varepsilon_0^2} r^2 dr + \int_R^\infty \frac{A^2 R^4}{4\varepsilon_0^2 r^4} r^2 dr \right]$

$= 2\pi \varepsilon_0 \left\{ \frac{A^2}{4\varepsilon_0^2} \frac{R^3}{3} + \frac{A^2 R^4}{4\varepsilon_0^2} \frac{1}{R} \right\}$

$= \frac{2\pi A^2}{3\varepsilon_0}$ *

Alternatively

$W = \frac{1}{2} \int \rho V \, d\tau = \frac{1}{2} \int \frac{A}{r} \cdot \frac{A}{2\varepsilon_0} (2R - r) \cdot 4\pi r^2 dr$

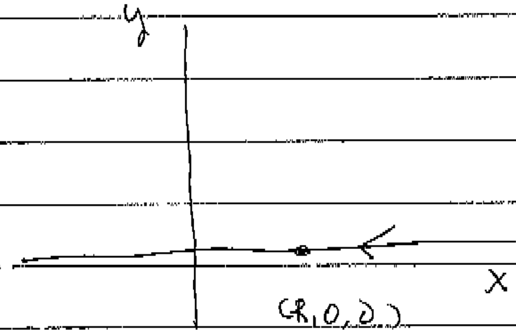
$= \frac{\pi A^2}{\varepsilon_0} \int_0^R r(2R - r) \, dr$

$= \frac{\pi A^2}{\varepsilon_0} \left[R r^2 - \frac{r^3}{3} \right]_0^R$

$= \frac{2\pi A^2}{3\varepsilon_0}$

Problem 2

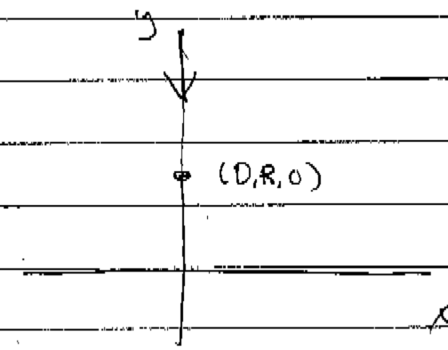
(a) Choose the simplest path.



$$V(R,0,0) = -\int_{\infty}^{(R,0,0)} \vec{E}\cdot d\vec{l}$$

$$= -\int_{\infty}^{(R,0,0)} E_x \, dx$$

$$= -\int_{\infty}^{(R,0,0)} \frac{3xy}{(x^2+y^2)^{5/2}} \, dx = 0 \quad (\because y=0)$$



$$V(0,R,0) = -\int_{\infty}^{(0,R,0)} \vec{E}\cdot d\vec{l}$$

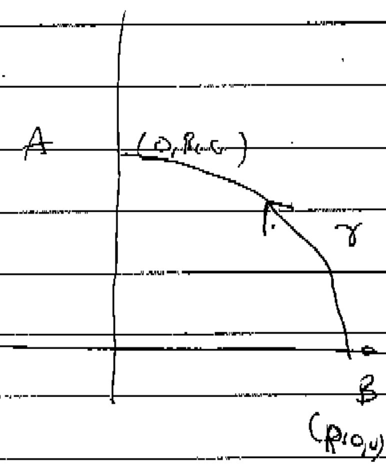
$$= -\int_{\infty}^{(0,R,0)} E_y \, dy$$

$$= -\int_{\infty}^{(0,R,0)} \frac{2y^2-x^2}{(x^2+y^2)^{5/2}} \, dy$$

$$= -\int_{\infty}^{(0,R,0)} \frac{2y^2}{y^5} \, dy \quad (\because x=0)$$

$$= \frac{1}{y^2}\Big|_{\infty}^{R} = \frac{1}{R^2} \;\#$$

(b) (i) $V_A - V_B = \frac{1}{R^2} - 0 = \frac{1}{R^2}$



(ii) $V_A - V_B = -\int_\gamma \vec{E} \cdot d\vec{l}$

$$\vec{l} = R(\cos\theta\, \hat{x} + \sin\theta\, \hat{y})$$

$$d\vec{l} = R(-\sin\theta\, \hat{x} + \cos\theta\, \hat{y})\, d\theta$$

$$\vec{E} \cdot d\vec{l} = R\, d\theta \left[E_x(-\sin\theta) + E_y \cos\theta \right]$$

$$= R\, d\theta \left[-\sin\theta \frac{3xy}{(x^2+y^2)^{5/2}} + \cos\theta \frac{2y^2 - x^2}{(x^2+y^2)^{5/2}} \right]$$

$$= R\, d\theta \left[-\sin\theta \frac{3\sin\theta\cos\theta}{R^3} + \cos\theta \frac{2\sin^2\theta - \cos^2\theta}{R^3} \right]$$

$$= -\frac{\cos\theta\, d\theta}{R^2}$$

$$V_A - V_B = -\int_0^{\pi/2} \left(-\frac{1}{R^2} \cos\theta\, d\theta \right) = \frac{1}{R^2} \sin\theta \Big|_0^{\pi/2} = \frac{1}{R^2}$$

#

Problem 3

(a) $$V = \frac{e(-e)}{4\pi\epsilon_0 a} + \frac{(e)(-e)}{4\pi\epsilon_0 a}$$

$$= \frac{-e^2}{2\pi\epsilon_0 a}$$

(b) $$V = \frac{e(+e)}{4\pi\epsilon_0(2a)} + \frac{e(+e)}{4\pi\epsilon_0(2a)}$$

$$= \frac{e^2}{4\pi\epsilon_0 a}$$

(c) $$V = \frac{1}{2}\left[\frac{-e^2}{2\pi\epsilon_0 a} + \frac{e^2}{4\pi\epsilon_0 a} \cdots\cdots\right]$$

$$= \frac{-e^2}{4\pi\epsilon_0 a}\left[1 - \frac{1}{2} + \frac{1}{3} \cdots\right]$$

$$= \frac{-e^2}{4\pi\epsilon_0 a}\ln(1+1) = \frac{-e^2}{4\pi\epsilon_0 a}\ln 2$$