PHY 801 HW#1 Solutions

1) $\left(1+\frac{\vec{\eta}\cdot\vec{\sigma}}{2}\right)\left(E-\vec{\sigma}\cdot\vec{p}\right)\left(1+\frac{\vec{\eta}\cdot\vec{\sigma}}{2}\right)$

$$= (E-\vec{\sigma}\cdot\vec{p}) + \frac{1}{2}\{\vec{\eta}\cdot\vec{\sigma},\, E-\vec{\sigma}\cdot\vec{p}\} + o(\eta^2)$$

$$= (E-\vec{\sigma}\cdot\vec{p}) + \frac{E}{2}\vec{\eta}\cdot\{\vec{\sigma},\mathbb{1}\} - \frac{\eta_i p_j}{2}\{\sigma_i,\sigma_j\} + o(\eta^2)$$

$$= (E-\vec{\sigma}\cdot\vec{p}) + E\,\vec{\eta}\cdot\vec{\sigma} - \delta_{ij}\eta^i p^j$$

$$= (E-\vec{\eta}\cdot\vec{p}) - \vec{\sigma}\cdot(\vec{p}-E\vec{p})$$

2) A 4-vector transforms as

$$A^\mu \to \Lambda^\mu{}_\nu A^\nu$$

In terms of $\vec{\varepsilon}$ and $\vec{\eta}$

$$A^0 \to A^0 - \vec{\eta}\cdot\vec{A}$$

$$\vec{A} \to \vec{A} - \vec{\eta}A^0 - \vec{\varepsilon}\times\vec{A}$$

Using the notation adopted in lectures

$$\psi \to V\psi \quad \text{and} \quad \chi \to (V^{-1})^\dagger \chi$$

where

$$V = \left(1 + \frac{i}{2}\vec{\varepsilon}\cdot\vec{\sigma} - \frac{1}{2}\vec{\eta}\cdot\vec{\sigma}\right)$$

First, Consider

$$\psi^\dagger \sigma^\mu \psi \qquad \sigma^\mu = (1, \vec{\sigma})$$

For $\mu = 0$ $\quad \psi^\dagger \sigma^0 \psi = \psi^\dagger \psi \to \psi^\dagger V^\dagger V \psi$

$$V^\dagger V = \left(1 - \tfrac{i}{2}\vec{\varepsilon}\cdot\vec{\sigma} - \tfrac{1}{2}\vec{\eta}\cdot\vec{\sigma}\right)\left(1 + \tfrac{i}{2}\vec{\varepsilon}\cdot\vec{\sigma} - \tfrac{1}{2}\vec{\eta}\cdot\vec{\sigma}\right)$$

$$= 1 - \vec{\eta}\cdot\vec{\sigma} \qquad \text{to } O(\varepsilon)\ O(\eta)$$

$$\therefore \psi^\dagger \sigma^0 \psi \to \psi^\dagger \sigma^0 \psi - \vec{\eta}\cdot \psi^\dagger \vec{\sigma} \psi$$

For $\mu = k \neq 0$ $\quad \psi^\dagger \sigma^\mu \psi = \psi^\dagger \sigma^k \psi \to \psi^\dagger V^\dagger \sigma^k V \psi$

Evaluate

$$V^\dagger \sigma^k V = \left(1 - \tfrac{i}{2}\vec{\varepsilon}\cdot\vec{\sigma} - \tfrac{1}{2}\vec{\eta}\cdot\vec{\sigma}\right)\sigma_k\left(1 + \tfrac{i}{2}\vec{\varepsilon}\cdot\vec{\sigma} - \tfrac{1}{2}\vec{\eta}\cdot\vec{\sigma}\right)$$

$$= \sigma_k + \tfrac{i}{2}\left[\sigma_k, \vec{\varepsilon}\cdot\vec{\sigma}\right] - \tfrac{1}{2}\left\{\vec{\eta}\cdot\vec{\sigma}, \sigma_k\right\}$$

$$\left[\sigma_k, \vec{\varepsilon}\cdot\vec{\sigma}\right] = \left[\sigma_k, \varepsilon_j \sigma_j\right] = 2i\, \epsilon_{kjl}\, \sigma_l\, \varepsilon_j$$

confusing notation!

$$= 2i\left(\vec{\varepsilon}\times\vec{\sigma}\right)_k$$

$$\left\{\vec{\eta}\cdot\vec{\sigma}, \sigma_k\right\} = \eta_j \left\{\sigma_j, \sigma_k\right\} = 2\eta_k$$

$$\therefore \psi^\dagger \vec{\sigma} \psi \to \psi^\dagger \vec{\sigma} \psi - \vec{\varepsilon} \times \psi^\dagger \vec{\sigma} \psi - \vec{\eta}\, \psi^\dagger \sigma^0 \psi$$

$\Rightarrow \psi^\dagger \sigma^\mu \psi$ transform as 4-vector

The transformation property of $\chi^\dagger \bar{\sigma}^\mu \chi$ can be worked out similarly so we can be brief

$$\chi^\dagger \bar{\sigma}^\mu \chi \to \chi^\dagger V^{-1} \bar{\sigma}^\mu (V^{-1})^\dagger \chi$$

For $\mu = 0$ $\quad \chi^\dagger \bar{\sigma}^0 \chi \to \chi^\dagger V^{-1} (V^{-1})^\dagger \chi$

$$V^{-1}(V^{-1})^\dagger = \left(1 - \tfrac{i}{2} \vec{\varepsilon}\cdot\vec{\sigma} + \tfrac{1}{2}\vec{\eta}\cdot\vec{\sigma}\right)\left(1 + \tfrac{i}{2}\vec{\varepsilon}\cdot\vec{\sigma} + \tfrac{1}{2}\vec{\eta}\cdot\vec{\sigma}\right)$$

$$= 1 + \vec{\eta}\cdot\vec{\sigma} \equiv 1 - \vec{\eta}\cdot\vec{\bar{\sigma}}$$

$$\therefore \chi^\dagger \bar{\sigma}^0 \chi \to \chi^\dagger \chi - \chi^\dagger(-\vec{\sigma})\chi$$

For $\mu = k \neq 0$ $\quad \chi^\dagger \bar{\sigma}^k \chi = -\chi^\dagger \sigma^k \chi$

$$\to -\chi^\dagger V^{-1} \sigma^k (V^{-1})^\dagger \chi$$

$$V^{-1}\sigma^k (V^{-1})^\dagger = \left(1 - \tfrac{i}{2}\vec{\varepsilon}\cdot\vec{\sigma} + \tfrac{1}{2}\vec{\eta}\cdot\vec{\sigma}\right)\sigma^k\left(1 + \tfrac{i}{2}\vec{\varepsilon}\cdot\vec{\sigma} + \tfrac{1}{2}\vec{\eta}\cdot\vec{\sigma}\right)$$

$$= \sigma^k - \varepsilon^{kij}\varepsilon_i \sigma_j + \eta^k$$

$$\therefore V^{-1}(-\vec{\sigma})(V^{-1})^\dagger \to V^{-1}(-\vec{\sigma})(V^{-1})^\dagger$$

$$- \vec{\varepsilon} \times V^{-1}(-\vec{\sigma})(V^{-1})^\dagger$$

$$- \vec{\eta}\cdot V^{-1}(-\vec{\sigma})(V^{-1})^\dagger$$

Hence $\chi^\dagger \bar{\sigma}^\mu \chi$ transforms as 4-vector

3) a) $$\xi \cdot \chi = \xi^a \chi_a$$

$$= \varepsilon^{ab} \xi_b \chi_a$$

$$= -\varepsilon^{ab} \chi_a \xi_b \qquad \text{Grassman}$$

$$= \varepsilon^{ba} \chi_a \xi_b \qquad \text{antisym of } \varepsilon$$

$$= \chi^b \xi_b$$

$$= \chi \cdot \xi$$

b) $$\chi_a \xi^a = \chi_a \varepsilon^{ab} \xi_b$$

$$= \varepsilon^{ab} \chi_a \xi_b$$

$$= -\varepsilon^{ba} \chi_a \xi_b \qquad \text{antisym of } \varepsilon$$

$$= -\chi^b \xi_b$$

$$= -\chi^a \xi_a$$

4) a) $$i\sigma_2 \chi \to i\sigma_2 \left(1 + i\vec{\varepsilon} \cdot \frac{\vec{\sigma}}{2} + \vec{\eta} \cdot \frac{\vec{\sigma}}{2}\right) \chi$$

$$= \left(1 - i\vec{\varepsilon} \cdot \frac{\vec{\sigma}^*}{2} - \vec{\eta} \cdot \frac{\vec{\sigma}^*}{2}\right) i\sigma_2 \chi$$

since $\sigma_2 \sigma_i = -\sigma_i^* \sigma_2$

$$= V^* (i\sigma_2 \chi)$$

b) $$(i\sigma_2 \chi)^T \xi \longrightarrow (V^* (i\sigma_2 \chi))^T (V^{-1})^\dagger \xi$$

$$= (i\sigma_2 \chi)^T \underbrace{V^\dagger (V^{-1})^\dagger}_{1} \xi = (i\sigma_2 \chi)^T \xi$$