PHYSICS 801 SUSY : Lecture 2

Coleman - Mandula Theorem

But SUSY was not discovered that way! (not banging the wall, trying to solve hierachy problem)

One of the dominant themes in 20 th century physics was that of symmetry.

Question : Have we exploited all kinds of symmetry consistent with Lorentz invariance?

Consider symmetry charge, say electric charge

$$Q = e \int d^3x \, \psi^\dagger \psi$$

or an SU(2) charge

$$T = g \int d^3x \, \psi^\dagger (\tau/2) \psi$$

both Lorentz scalars

$$\therefore \quad Q|J\rangle = |\text{same } J, \text{ possibly different quantum \#}\rangle$$

Can we construct a tensorial charge?

Examples of

- Vector charge P_μ, momentum
- ~~[illegible]~~ Antisymmetric tensor $M_{\mu\nu}$ angular momentum

Can we construct say a ~~[illegible]~~ conserved symmetric tensor charge $Q_{\mu\nu}$

Coleman ~~&~~ Mandula showed that it is <u>not</u> possible
(State Theorem Here)

A simplified argument due to Witten

Consider $\quad Q_{\mu\nu}|p\rangle = \underbrace{(\alpha p_\mu p_\nu + \beta g_{\mu\nu})}_{\text{more general form taken from the tensors of our disposal}}|p\rangle$

$|p\rangle$: single particle state

Consider a two-particle state, assuming that $Q_{\mu\nu}$ are addictive, conserved, and act on only one particle at a time

$$Q_{\mu\nu}|p^{(1)}, p^{(2)}\rangle = \left[\alpha\,(p^1_\mu p^1_\nu + p^2_\mu p^2_\nu) + 2\beta g_{\mu\nu}\right]|p^1, p^2\rangle$$

In elastic scattering $1+2 \to 3+4$, conservation of eigenvalues

$$\Rightarrow \quad p^1_\mu p^1_\nu + p^2_\mu p^2_\nu = p^3_\mu p^3_\nu + p^4_\mu p^4_\nu$$

On the other hand, 4-momentum conservation

$$\Rightarrow \quad p_\mu^1 + p_\mu^2 = p_\mu^3 + p_\mu^4$$

Only simultaneous solutions are

$p_\mu^{(1)} = p_\mu^{(3)} \qquad p_\mu^{(2)} = p_\mu^{(4)}$

or $\quad p_\mu^1 = p_\mu^4 \qquad p_\mu^2 = p_\mu^3$

only forward and backward scattering can occur

Seems to No room for further conserved operators

P_μ & $M_{\mu\nu}$ leaves a scattering angle unfixed.

Existence of additional tensorial charge determines the scattering angle to some discrete values

Coleman-Mandula Thm

The Lie algebra containing Poincare ~~symmetry~~ Algebra and any Lie algebra ~~~~ ~~by~~ defined by the generators T^a is a direct sum

$$[T^a, T^b] = i t^{ab}_c T^c \qquad t^{ab}_c \text{ - structure constants}$$

$$[T^a, P_\mu] = [T^a, M_{\mu\nu}] = 0$$

There had been some suggestions at that time that the Poincare symmetry + internal symmetries may interwine and belong to a larger group like SU(6))

$\boxed{\text{Loophole}}$ → spinorial charge Q_a

↑ spinor component (more precise about that later)

$$Q_a |J\rangle = |J \pm 1/2\rangle$$

Q_a does not contribute to matrix element (particle spin remains in 2→2 scattering) ⇒ no-go thm not apply

Is it possible to extend Poincare algebra to include those spinorial charges Q_a ?

Yes! - Gol'fand & Likhtman (71)

- Haag, Lopuszanski, & Sohnius (75) most general such "SUSY algebra"

History does not follow a logical path.

- Wess & Zumino constructed the first linearly realized SUSY theories in 4D ('74) without the knowledge of these works

- SUSY first made its appearance in string field theory theory by ~~Neveu~~ Schwarz ('71) & Ramond (71) before Wess-Zumino, though the construction is 2D.

(2D SUSY & 4D SUSY are related via GSO projection)

SUSY algebra is a set of commutation relations between the symmetry charges.

Since Q_a is a spinor, algebra involves both commutation and anticommutation relations

$\Rightarrow$ a <u>graded</u> algebra

How does the algebra look like?

$$[Q_a, H] = 0$$ Q_a symmetry operator

$$[\underbrace{\{Q_a, Q_b\}}_{\text{symmetric product of 2 spin 1/2 fields}}, H] = 0$$

$\rightarrow$ spin 1

Guess $\{Q_a, Q_b\} \sim P_\mu$ ← spin 1 and commute with Hamiltonian.

indices don't match

Further guess $\{Q_a, Q_b\} \sim \gamma^\mu P_\mu$ qualitatively right (more precise later)

<u>A radical idea !</u>

Q_a is the square root of space-time derivative

$\sqrt{KG} \Longleftrightarrow$ Dirac. $\sqrt{\text{Dirac}} \Leftrightarrow$ SUSY

$\Rightarrow$ Extending the concept of <u>spacetime by</u> introducing <u>fermionic dimensions</u> $\sim \sqrt{\text{spacetime}}$

$\boxed{\text{like } \sqrt{-1} = i}$

Consequences of SUSY algebra

$$\{Q_a, \bar{Q}_b\} = 2\gamma^\mu_{ab} P_\mu \qquad \bar{Q}_b = (Q^T\gamma^0)_b$$

$$[Q_a, P_\mu] = 0$$

$$[Q_a, M_{\mu\nu}] = \frac{1}{2}(\sigma_{\mu\nu})_{ab} Q_b \qquad \sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$$

Will derive this algebra more carefully later.

Here we investigate some consequences:

① First $\quad [Q_a, P_\mu P^\mu] = 0$

Consider two states related to each other by

$$Q_a \underset{\substack{\uparrow \\ \text{boson}}}{|b\rangle} = |f\rangle$$

$$P_\mu P^\mu Q_a |b\rangle = Q_a \underbrace{P_\mu P^\mu |b\rangle}_{m_b^2 |b\rangle} = m_b^2 \underbrace{Q_a |b\rangle}_{|f\rangle}$$

$$\|$$

$$P_\mu P^\mu |f\rangle$$

$$\|$$

$$m_f^2 |f\rangle$$

Hence $\boxed{m_b = m_f}$ Bose-Fermi degeneracy

2 $$\{Q_a, \bar{Q}_b\} = 2\gamma^\mu_{ab} P_\mu$$

$$\{Q_a, Q_b\}\gamma^0_{bc} = 2\gamma^\mu_{ac} P_\mu$$

Contracting with γ^0_{ca}

$$\sum_{a,b} \{Q_a, Q_b\} \underbrace{(\gamma^0)^2_{ba}}_{\Downarrow \; \mathbb{1}} = 2 \underbrace{\underbrace{\gamma^\mu_{ac}\gamma^0_{ca}}_{\mathrm{tr}(\gamma^\mu\gamma^0)} }_{4g^{0\mu}} P_\mu$$

$$\Rightarrow \quad \sum_a Q_a^2 = 4P^0 = 4H$$

$$\Rightarrow \quad H = \frac{1}{4}\sum_a Q_a^2 \geq 0$$

[Note: holds for <u>global</u> SUSY, not SUGRA]

SUSY Breaking

SUSY vacuum $\Leftrightarrow$ $\langle H \rangle = 0$

"$\Rightarrow$" $Q_a |\Omega\rangle = 0$

$$\Rightarrow H = \frac{1}{4} \sum_a Q_a^2 |\Omega\rangle = 0$$

"$\Leftarrow$" $\langle \Omega | H | \Omega \rangle = 0$

$$\Rightarrow \frac{1}{4} \sum_a \langle \Omega | Q_a Q_a | \Omega \rangle = 0$$

$$\Rightarrow \frac{1}{4} \sum_a \| Q_a |\Omega\rangle \|^2 = 0$$

$$\Rightarrow Q_a |\Omega\rangle = 0$$

Hence $\langle \Omega | H | \Omega \rangle \neq 0$ is an order parameter for global ~~SUSY~~

To clarify, we have 4 cases

① $\langle \phi \rangle = 0$ $\quad V(\langle \phi \rangle) = 0$ gauge sym & SUSY unbroken

② $\langle \phi \rangle = 0$ $\quad V(\langle \phi \rangle) \neq 0$ gauge sym unbroken, SUSY spontaneously broken

③ $\langle \phi \rangle \neq 0$ $\quad V(\langle \phi \rangle) = 0$ ~~gauge~~ sym but SUSY not broken

④ $\langle \phi \rangle \neq 0$ $\quad V(\langle \phi \rangle) \neq 0$ both SUSY & gauge sym broken

- A consequence of $\langle\Omega|H|\Omega\rangle$ being an order parameter is that SUSY can be broken at <u>finite volume</u>.

- In SUGRA, one can have ~~SUSY~~ but $\langle\Omega|H|\Omega\rangle = 0$

- Bose - Fermi pairing

$$Q_a|b\rangle = |f\rangle$$

all states are paired except ground state

$$Q_a|\Omega\rangle = 0$$

When SUSY is broken, all states including ground states are paired. How precisely?

Massless goldstone fermions (Goldstino) when SUSY is spontaneously broken.

Using the Goldstino $\tilde{G}$

$$|b\rangle, \quad |b\tilde{G}\uparrow\rangle \quad |b\tilde{G}\downarrow\rangle \qquad |b\tilde{G}\uparrow\tilde{G}\downarrow\rangle$$

$\uparrow$ bosons   fermions   fermion   boson

Witten Index

$$I = n_B^{E=0} - n_F^{E=0}$$

$$= \mathrm{Tr}\left((-1)^F\right)$$

$(-1)^F = 1$ boson

$\quad\quad\;\; -1$ fermion

a topological quantity

$I = 0$ is a necessary but not sufficient condition for ~~SUSY~~