

Matching the circular Wilson loop with dual open string solution at 1-loop in strong coupling

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arXiv:0803.0315 [hep-th]

- Compute the 1-loop correction to the effective action for the string solution ending on a straight line at the boundary.

- Compute the 1-loop correction to the effective action for the string solution in $AdS_5 \times S^5$ dual to the circular Wilson loop.

- More generically, the method we use can be applied whenever the two dimensional spectral problem factorizes, to regularize and define the fluctuation determinants in terms of solutions of one-dimensional differential equations.

- It can be applied to non-homogeneous solutions both for open and closed strings and to various boundary conditions.

- Circular Wilson loop, 1-loop partition function result matches, up to a factor of two, the expectation from the exact gauge theory computation.

The discrepancy can be attributed to an overall constant in the string partition function coming from the measure, which we have not fixed.

Wilson loop solutions

Checking AdS/CFT:

- Matching the anomalous dimension of certain operators in the gauge theory to the energy of a corresponding closed string in $AdS_5 \times S^5$.
- Comparing expectation value of Wilson loops with the string partition function of a dual string solution that at the boundary of AdS ends on the loop.

$$\langle W \rangle = Z$$

$$W = \frac{1}{N} \text{Tr} P \exp \left(\oint (iA_\mu \dot{x}^\mu + \Phi_i |\dot{x}| \theta^i) ds \right)$$

- Duality should be true to all orders in $1/N$ expansion and all orders in gauge theory coupling g^2 , and on string theory side to full quantum string and all orders in string coupling.

$$\lambda = g^2 N, \quad 4\pi g_s = g^2$$

Planar level $N = \infty$, free string $g_s = 0$ and test the correspondence as a function of λ . Gauge theory weakly coupled at small λ . String theory is weakly coupled at large λ . Hard to check correspondence in general.

Wilson Loops constructed in N. Drukker, D. J. Gross and H. Ooguri, [arXiv:hep-th/9904191]

These are loops at the boundary of AdS_5 .

- Single straight line - globally supersymmetric BPS object.

- Parallel lines separated by a length L . Related to the computation of quark-antiquark potential.

- Cusp Wilson loop –discussed recently in connection with scattering amplitudes of gluons in L. F. Alday and J. M. Maldacena, “Gluon scattering amplitudes at strong coupling,” [arXiv:0705.0303 [hep-th]]

- Circular Wilson loop – not invariant under all conformal transformations

On string theory side, string solutions minimize the area bounded by these loops at the boundary of AdS_5 .

1-loop string corrections to effective action.

For straight string we should get zero for the 1-loop effective action.

For circular string we expect from gauge theory

$$\Gamma = -\sqrt{\lambda} + \frac{3}{4} \ln \lambda + \frac{1}{2} \ln \frac{\pi}{2} + \frac{3}{8} \frac{1}{\sqrt{\lambda}} + \dots$$

It was shown in J. K. Erickson, G. W. Semenoff and K. Zarembo, [arXiv:hep-th/0003055] that the expectation value of circular Wilson loop computed exactly at planar level and all orders in λ . The proposed result is

$$\langle W \rangle = \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda})$$

Computation can be expressed in terms of a Gaussian matrix model. Gauge theory computation extended to all orders in $1/N$ expansion in N. Drukker and D. J. Gross, [arXiv:hep-th/0010274] This was fully checked recently by direct gauge theory computation in V. Pestun, arXiv:0712.2824 [hep-th]

- Goal: check the gauge theory expression at strong coupling against the string theory beyond classical level in string theory. We need to compute 1-loop corrections to the effective action.

Straight string solution

AdS_5 metric

$$ds^2 = \frac{1}{z^2}(dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2 + dz^2)$$

solution is

$$x_0 = \tau, \quad z = \sigma \quad 0 \leq z < \infty$$

For τ we take a large interval $0 \leq \tau < 2\pi T$, T large.

Induced metric is AdS_2

$$ds_2^2 = \frac{1}{\sigma^2}(d\tau^2 + d\sigma^2)$$

with $2d$ curvature $R^{(2)} = -2$.

The classical action is

$$S = \sqrt{\lambda} \frac{T}{\epsilon}$$

Action actually is singular so we introduced a cutoff ϵ at small z . Linear divergency is proportional to the length of the Wilson loop

- Need to regularize it to get physical result

Consider Lagrange transform that introduces a boundary term making area finite.

Action is proportional to the volume part of the Euler number

$$S = -\sqrt{\lambda} \chi_v, \quad \chi_v = \frac{1}{4\pi} \int_M d^2\sigma \sqrt{g} R^{(2)} = -\frac{T}{\epsilon}$$

Natural topological way to regularize is to add a term proportional to boundary part of Euler number

$$\chi_b = \frac{1}{2\pi} \int_{\partial M} ds \kappa_g = \frac{T}{\epsilon}$$

making Euler number finite and integer. Regularized action is

$$S = -\sqrt{\lambda} \chi, \quad \chi = \chi_v + \chi_b = 0$$

One loop correction to the effective action

GS string in $AdS_5 \times S^5$. Bosonic part

$$S = \frac{\sqrt{\lambda}}{4\pi} \int d^2\sigma \sqrt{g} g^{ij} G_{\mu\nu}(x) \partial_i x^\mu \partial_j x^\nu$$

and quadratic fermionic part

$$S_F = \frac{\sqrt{\lambda}}{2\pi} \int d^2\sigma L_{2F}$$

$$L_{2F} = -i(\sqrt{g} g^{ij} \delta^{IJ} - \epsilon^{ij} s^{IJ}) \bar{\theta}^I \rho_i D_j \theta^J$$

where

$$\rho_i = \Gamma_A e_i^A$$

$$D_i \theta^I = \delta^{IJ} \nabla_i \theta^J - \frac{1}{2} i \epsilon^{IJ} \rho_i \theta^J, \quad \nabla_i = \partial_i + \frac{1}{4} \Omega_i^{AB} \Gamma_{AB}$$

Consider fluctuations near a particular solution
 N. Drukker, D. J. Gross and A. A. Tseytlin, [arXiv:hep-th/0001204]

- shown that the 1-loop effective action is finite for any string solution and any background metric

We fix background metric the induced metric

$$g_{ij} = h_{ij} = \frac{1}{\sigma^2} \delta_{ij}$$

Transverse bosonic string fluctuation action

$$S = \frac{\sqrt{\lambda}}{4\pi} \int d\tau d\sigma \frac{1}{\sigma^2} \left[\sigma^2 \partial_i \zeta^A \partial_j \zeta^A + 2(\zeta^1)^2 + 2(\zeta^2)^2 + 2(\zeta^3)^2 \right]$$

three bosonic fluctuations with mass squared = 2, and five with mass squared = 0

Longitudinal fluctuation Lagrangian the same as that for conformal ghosts.

Spectral problem needed to solve is

$$Lf = \Lambda f, \quad L = \sigma^2 (-\partial_0^2 - \partial_1^2) + 2 = -\nabla^2 + 2$$

- Boundary conditions. Since we want to compare the results between straight and circular string solutions we choose periodic boundary condition in τ . In σ we choose Dirichlet boundary conditions.

With $f(\tau, \sigma) = \sum_n g_n(\sigma) e^{im\tau}$ with $m = \frac{n}{T}$ then determinant of the operator is

$$\det L = \prod_m \det \left(\sigma^2 (-\partial_1^2 + m^2) + 2 \right)$$

T large, at the end replace sum by integral

Fermionic quadratic Lagrangian for straight string solution is

$$L_{2F} = -2i\sqrt{g}\bar{\theta} D_F \theta$$

$$D_F = -\sigma\Gamma_0\partial_0 + \sigma\Gamma_4\partial_1 - \frac{1}{2}\Gamma_4 + i\Gamma_0\Gamma_4$$

Can choose representation for Gamma matrices: $\Gamma_0 = i\sigma_2 \times I_8, \Gamma_4 = \sigma_1 \times I_8$, then Γ_0, Γ_4 play the role of worldsheet Dirac matrices.

Squaring Dirac operator we obtain the spectral problem

$$L_F \theta = \Lambda \theta$$

$$L_F = -\nabla_i \nabla^i + \frac{R^{(2)}}{4} + 1 = \sigma^2 (-\partial_1^2 + m^2) + \frac{3}{4} + \Gamma_{04} m \sigma$$

We have eight fermions with mass squared = 1

Putting together bosons and fermions we obtain 1-loop partition function

$$Z = \frac{\det^{8/2}(-\nabla^2 + \frac{R^{(2)}}{4} + 1)}{\det^{3/2}(-\nabla^2 + 2) \det^{5/2}(-\nabla^2)}$$

Computation of functional determinants difficult in general. When they reduce to one-dimensional operators can use a nice method to compute ratio of determinants.

Computation of ratio of determinants

Method found long ago in

I.M. Gelfand and A.M. Yaglom, J. Math. Phys., **1**:
48-69,1960

For two operators defined in an interval $x \in [a, b]$
and with Dirichlet boundary conditions

$$L = -P_0(\sigma) \frac{d^2}{d\sigma^2} + P_1(\sigma) \frac{d}{d\sigma} + P_2(\sigma)$$

$$\hat{L} = -P_0(\sigma) \frac{d^2}{d\sigma^2} + \hat{P}_1(\sigma) \frac{d}{d\sigma} + \hat{P}_2(\sigma)$$

the ratio of determinants can be computed as

$$\frac{\det L}{\det \hat{L}} = \frac{e^{-\frac{1}{2} \int_a^b d\sigma P_1(\sigma) P_0^{-1}(\sigma)} \psi(b)}{e^{-\frac{1}{2} \int_a^b d\sigma \hat{P}_1(\sigma) P_0^{-1}(\sigma)} \hat{\psi}(b)}$$

where ψ and $\hat{\psi}$ are solutions of the initial value
problems

$$L\psi = 0, \quad \hat{L}\hat{\psi} = 0, \quad \psi(a) = \hat{\psi}(a) = 0, \quad \psi'(a) = \hat{\psi}'(a) = 1$$

For $P_1(\sigma) = \hat{P}_1(\sigma) = 0$ this reduces to

$$\frac{\det L}{\det \hat{L}} = \frac{\psi(b)}{\hat{\psi}(b)}$$

- can be generalized to any boundary conditions.
- can have different boundary conditions at the two ends
- both operators must have the same boundary conditions
- needs modification if operators have zero modes

Return to the determinants of interest and compute ratio. 1-loop effective action is

$$\Gamma_1 = \frac{1}{2} \ln \prod_m P_m$$

$$P_m = \frac{\det^3[-\partial_1^2 + m^2 + \frac{2}{\sigma^2}] \det^5[-\partial_1^2 + m^2]}{\det^4[-\partial_1^2 + m^2 + \frac{3}{4\sigma^2} + \frac{m}{\sigma}] \det^4[-\partial_1^2 + m^2 + \frac{3}{4\sigma^2} - \frac{m}{\sigma}]} \quad (1)$$

Singularity at $\sigma = 0$, introduce a cutoff ϵ . Singularity is reflected in the initial value solutions.

• We take finite interval $\sigma \in [\epsilon, R]$, and at the end take the limit $R \rightarrow \infty$. It is crucial that the final result is independent of R . After the limit $R \rightarrow \infty$ at the very end of the computation we take $\epsilon \rightarrow 0$.

Initial value solutions: transversal bosonic operator

$$-g'' + \left(m^2 + \frac{2}{\sigma^2}\right)g = 0, \quad g(\epsilon) = 0, \quad g'(\epsilon) = 1$$

solution

$$g(\sigma) = \frac{1}{m^3 \epsilon \sigma} \left[m(\sigma - \epsilon) \cosh m(\sigma - \epsilon) - (1 - \epsilon m^2 \sigma) \sinh m(\sigma - \epsilon) \right]$$

For fermions we need

$$\left[-\partial_1^2 + m^2 + \frac{3}{4\sigma^2} + \frac{m}{\sigma} \right] \theta = 0$$

$$\theta(\sigma) = \frac{1}{4m^2 \sqrt{\epsilon \sigma}} \left[(2m\sigma - 1)e^{m(\sigma - \epsilon)} - (2m\epsilon - 1)e^{-m(\sigma - \epsilon)} \right]$$

Solutions blow up when boundary is at $\epsilon = 0$. We expect to have $1/\epsilon$ divergency also at 1-loop order.

For the free bosons

$$-g'' + m^2 g = 0, \quad g = \frac{1}{m} \sinh m(\sigma - \epsilon)$$

The ratio of determinants needed are (for large R)

$$\frac{\det[-\partial_1^2 + m^2 + \frac{2}{\sigma^2}]}{\det[-\partial_1^2 + m^2 + \frac{3}{4\sigma^2} + \frac{m}{\sigma}]} = \frac{m\epsilon + 1}{m\sqrt{\epsilon R}}$$

$$\frac{\det[-\partial_1^2 + m^2]}{\det[-\partial_1^2 + m^2 + \frac{3}{4\sigma^2} + \frac{m}{\sigma}]} = \sqrt{\frac{\epsilon}{R}}$$

$$\frac{\det[-\partial_1^2 + m^2]}{\det[-\partial_1^2 + m^2 + \frac{3}{4\sigma^2} - \frac{m}{\sigma}]} = \frac{2m\sqrt{R\epsilon}}{2m\epsilon + 1}$$

In partition function R -dependence goes away. We get

$$\Gamma_1 = \frac{T}{\epsilon} \left(1 + \ln \frac{\epsilon}{4T}\right)$$

A mechanism should cancel the divergency left and physical result

$$\Gamma_1 = 0$$

Circular Wilson loop solution

AdS_5 metric in polar coordinates

$$ds^2 = \frac{1}{z^2}(dr^2 + r^2 d\phi^2 + dz^2 + dx_i^2)$$

String solution ending on a circular loop of radius a at the boundary

$$z = \sqrt{a^2 - r^2}, \quad 0 \leq r \leq a, \quad 0 \leq \phi < 2\pi$$

Physical result should not depend on radius a . Minimal surface has the topology of a disk. To compute the area we introduce again a cutoff at $z = \epsilon$

$$S = -\sqrt{\lambda} + \sqrt{\lambda} \frac{a}{\epsilon} = -\sqrt{\lambda} \chi_v$$

Completing the Euler number by adding a boundary term proportional to the boundary part Euler number get regularized action

$$S = -\sqrt{\lambda} \chi = -\sqrt{\lambda}$$

Induced metric again AdS_2

$$ds_2^2 = \frac{1}{\sinh^2 \sigma} (d\sigma^2 + d\tau^2)$$

Solution in Polyakov formulation in conformal gauge is

$$r = \frac{a}{\cosh \sigma}, \quad z = a \tanh \sigma, \quad 0 \leq \sigma < \infty, \quad 0 \leq \tau \equiv \phi < 2\pi$$

Cutoff in $z = \epsilon$ translates in cutoff in $\sigma = \epsilon_0$, related by $\epsilon = a \tanh \epsilon_0$.

1-loop correction to the effective action

Bosonic quadratic fluctuation Lagrangian

$$\begin{aligned} S = & \frac{\sqrt{\lambda}}{4\pi} \int d\tau d\sigma \frac{1}{s^2} [s^2 (\partial_0 \zeta^A)^2 + s^2 (\partial_1 \zeta^A)^2 \\ & + 2 \left((\zeta^2)^2 + (\zeta^3)^2 + (\zeta^4)^2 \right) + (s^2 + 2) \left((\zeta^0)^2 + (\zeta^1)^2 \right) \\ & - 2sc \dot{\zeta}^0 \zeta^1 + 2sc \zeta^0 \dot{\zeta}^1] \end{aligned}$$

where $s = \sinh \sigma$, $c = \cosh \sigma$.

Lagrangian of the two coupled modes is the same as ghost Lagrangian. Remaining transversal fluctuations have mass squared = 2. Spectral problem is

$$Lf = \Lambda f, \quad L = \sinh^2 \sigma (-\partial_0^2 - \partial_1^2) + 2 = -\nabla^2 + 2$$

• τ is periodic, ansatz $f(\sigma, \tau) = e^{im\tau} g(\sigma)$, so m has integer values. Range in $\sigma \in [\epsilon_0, R]$. In σ we take Dirichlet boundary conditions at both ends.

Determinant of operator L can be written as

$$\det L = \prod_{m=-\infty}^{\infty} \det \left(\sinh^2 \sigma (-\partial_1^2 + m^2) + 2 \right)$$

Compute the Fermionic Lagrangian

$$L_{2F} = -2i\sqrt{g}\bar{\Psi}D_F\Psi$$

$$D_F = -\sinh \sigma \Gamma_0 \partial_0 + \sinh \sigma \Gamma_4 \partial_1 - \frac{1}{2} \cosh \sigma \Gamma_4 + i\Gamma_0 \Gamma_4$$

For small σ this is the same as for straight string.

The resulting spectral problem for the fermions is

$$L_F \theta = \Lambda \theta, \quad L_F = -\nabla_i \nabla^i + \frac{R^{(2)}}{4} + 1$$

$$L_F = \sinh^2 \sigma (-\partial_1^2 + r^2) + \frac{3}{4} + \frac{\sinh^2 \sigma}{4} + \Gamma_{04} r \cosh \sigma \sinh \sigma$$

Fermions here are anti-periodic so summation indices r is half-integer. We transform the fermionic sum to sum over integers using some supersymmetric shifts.

Computation of ratio of determinants can be done analytically. Initial value problems have relatively simple solutions. We get the ratio of determinants

$$\begin{aligned} & \frac{\det[-\partial_1^2 + m^2 + \frac{2}{\sinh^2 \sigma}]}{\det[-\partial_1^2 + (m - \frac{1}{2})^2 + \frac{1}{4} + \frac{3}{4 \sinh^2 \sigma} + (m - \frac{1}{2}) \coth \sigma]} \\ &= \sqrt{2 \sinh \epsilon_0} \frac{m + \coth \epsilon_0}{m + 1} e^{-\frac{\epsilon_0}{2}} \end{aligned}$$

$$\begin{aligned} & \frac{\det[-\partial_1^2 + m^2]}{\det[-\partial_1^2 + (m - \frac{1}{2})^2 + \frac{1}{4} + \frac{3}{4 \sinh^2 \sigma} + (m - \frac{1}{2}) \coth \sigma]} \\ &= \sqrt{2 \sinh \epsilon_0} e^{-\frac{\epsilon_0}{2}} \end{aligned}$$

$$\frac{\det[-\partial_1^2 + m^2]}{\det[-\partial_1^2 + (m + \frac{1}{2})^2 + \frac{1}{4} + \frac{3}{4 \sinh^2 \sigma} - (m + \frac{1}{2}) \coth \sigma]}$$

$$= \frac{\sqrt{2 \sinh \epsilon_0}}{\cosh \epsilon_0} \frac{m + 1}{1 + (2m + 1) \tanh \epsilon_0} e^{\frac{\epsilon_0}{2}}$$

Put them together get 1-loop correction result

$$\Gamma_1 = \frac{a}{\epsilon} \left(1 + \ln \frac{\epsilon}{4a}\right) + \frac{1}{2} \ln(2\pi)$$

- The finite part of the 1-loop effective action is independent of the radius of the circle.
- The ($\epsilon \rightarrow 0$) divergent part is the same as the one for a straight string of length $T = a$. If we subtract both the result is finite:

$$\Gamma_1 = \frac{1}{2} \ln(2\pi)$$

Likely subtraction procedure works at higher orders in strong coupling expansion.

This is to be compared to the gauge theory expectation at 1-loop

$$\Gamma_1 = \frac{1}{2} \ln \frac{\pi}{2}$$

Our result means that for circular loop

$$Z = \frac{1}{2} \langle W \rangle$$

- could be due to fixing the relative normalization between of operators in gauge theory and supergravity fields in AdS_5 .
- this factor might come from the string measure

Final Remarks

- We computed 1-loop effective action for the string Wilson loop solution. Open question: fixing the discrepancy factor of 2 between string partition function and expectation value of Wilson loop

- We computed also 1-loop correction for circular Wilson loop solution wrapped k -times. Result is

$$\Gamma_1 = \frac{1}{2}[\ln(2\pi) + (4k + 1) \ln k - 2 \ln \Gamma(1 + k)]$$

Important open question. Check matching with gauge theory result for $k \neq 1$.

- 2-loop string computation very interesting to compare to gauge theory.