Accelerations in Special Relativity

Except for the twin paradox, we’ve avoided situations where accelerations occur. We have also mentioned several times the equivalence between acceleration and a uniform gravitational field, and that the theory of general relativity is required to understand gravity. Putting these together, one is tempted to conclude that the theory of special relativity cannot handle situations where there are forces and accelerations. This is not true.

It turns out that the proper relativistic generalization of Newton’s law $\frac{d\vec{v}}{dt} = \vec{a}$ is

$$d\gamma \vec{v} \over dt = \vec{a} \tag{1}$$

where $\gamma = (1 - v^2/c^2)^{-1/2}$ and $\vec{a}$ is the acceleration due to some applied force. It is not obvious that this is so, and in a slightly more advanced class we would prove it. We can argue that it makes sense, however, by realizing that when the velocity is nearly $c$, the generalized Newton’s law shows that accelerations will increase $\gamma v$ to large values, keeping $v < c$ as needed. In other words, accelerations increase $\gamma v$, not just $v$. In fact, when the velocity becomes nearly $c$ (ultra-relativistic limit), Newton’s law becomes approximately (in 1-D)

$$c \frac{d\gamma}{dt} \approx a \tag{2}$$

so that all the acceleration does is increase $\gamma$.

Let us analyze the case of a particle moving in a uniform 1-D force field $\vec{a} = g \hat{x}$, with $v = 0$ at $t = 0$. We can integrate over $t$ to find

$$\gamma v = at \tag{3}$$

or, solving for $v$,

$$v = \frac{at}{\sqrt{1 + (at/c)^2}}; \quad \gamma = \sqrt{1 + (at/c)^2} \tag{4}$$

For small values of $t$, we get $v = at$, while for large values of $t$, we get $v \approx c$ and $\gamma = at/c \gg 1$, as expected. But we have avoided a very interesting question, namely how does the particle’s time (proper time $\tau$) relate to $t$? To figure this out, we note that at any given time during the acceleration we could go to an inertial frame moving at velocity $v$. We know that the particle’s clock will be ticking slower than we measure, so a little increment of time in our frame will be related to a little increment of proper time via $dt = \gamma d\tau$. You can check that the solution to this is

$$t = \frac{c}{a} \sinh (a\tau/c) = \frac{c}{2a} \left( e^{a\tau/c} - e^{-a\tau/c} \right) \tag{5}$$

Suppose we accelerate an astronaut at $a = 1g$ for 5 years. Plugging in numbers, $at/c = 5.15$, so that $v = 0.98c$, and $\gamma = 5.25$. How old is the astronaut? $\tau = (c/a) \sinh^{-1} (at/c) = 2.27$ years! After 10 years, her age is only 2.94 years.

A second acceleration problem that is easy to solve is the motion of a charged particle in a uniform magnetic field. Here the acceleration is

$$\vec{a} = \frac{q}{m} \vec{v} \times \vec{B} \tag{6}$$
The acceleration is perpendicular to the velocity, so the speed doesn’t change. Thus the particle undergoes uniform circular motion with radius

\[ r = \frac{m\gamma v}{qB} \]  

(7)

A third example is a charge being accelerated by a uniform electric field \( \vec{E} = E\hat{y} \) perpendicular to its initial velocity \( \vec{v} = v_0\hat{x} \). Newton’s law is

\[ \frac{d\gamma \vec{v}}{dt} = \frac{qE}{m} \]  

(8)

or, separating into components,

\[ \frac{d\gamma v_x}{dt} = 0; \quad \frac{d\gamma v_y}{dt} = \frac{qE}{m} \equiv a_y \]  

(9)

These equations are integrated to give

\[ \gamma v_x = \gamma_0 v_0; \quad \gamma v_y = a_y t \]  

(10)

so that the direction the particle is going is

\[ \theta(t) = \arctan \frac{v_y}{v_x} = \arctan \frac{a_y t}{\gamma_0 v_0} \]  

(11)

The velocity is

\[ \vec{v} = \frac{(\gamma_0 v_0, a_y t)}{\sqrt{\gamma_0^2 + (a_y t/c)^2}} \]  

(12)

Notice that, even though there is no \( \hat{x} \)-component of the force, the \( \hat{x} \)-component of the velocity decreases with time. This is because

\[ \gamma(t) = \sqrt{\gamma_0^2 + (a_y t/c)^2} \]  

(13)

is increasing with time and it is the produce \( \gamma v_x \) that must be constant in the absence of any force in the \( \hat{x} \) direction.

A plot of the motion is shown on the next page.
Figure 1: Motion of a relativistic particle in a uniform electric field perpendicular to the initial velocity.