As a 1-D model of a fiber, we take

\[ n = \begin{cases} 
    n_1 & |x| < a \\
    n_2 & |x| > a
\end{cases} \]  

(1)

We assume \( E(x, y, z) = E(x)e^{ikz} \), so the wave equation gives

\[ \frac{\partial^2 E(x)}{\partial x^2} + (k^2 - k_z^2)E(x) = 0 \]  

(2)

We are looking for solutions which oscillate for \( |x| < a \), and are exponentially damped for \( |x| > a \). Therefore it is convenient to define

\[ k_x = \sqrt{k_1^2 - k_z^2} \quad |x| < a \]
\[ \kappa = \sqrt{k_z^2 - k_2^2} \quad |x| > a \]  

(3)

Notice that \( k_x^2 + \kappa^2 = (n_1^2 - n_2^2)k_z^2 = V^2/a^2 \) where \( V \) is the V-number of the fiber. Then the desired solutions have the form

\[ E_1 = A \cos k_x x + B \sin k_x x \quad |x| < a \]
\[ E_2 = Ce^{-\kappa x} \quad |x| > a \]  

(4)

The boundary conditions are \( E_1(a) = E_2(a) \) and \( E_1'(a) = E_2'(a) \). For the even solutions, with \( B = 0 \), taking the ratio of the boundary conditions gives

\[ k_x a \tan k_x a = \kappa a = \sqrt{V^2 - k_x^2 a^2} \]  

(5)

The odd solutions, with \( A = 0 \) give similarly

\[ -k_x a \cot k_x a = \sqrt{V^2 - k_x^2 a^2} \]  

(6)
Figure 1: Graphical determination of mode frequencies. The number of intersection points determines the number of modes supported by the fiber.

The solutions can be found graphically or numerically as the intersection of the left and right hand sides of these equations. This is shown in the figure, where the red and blue show the left hand sides, and the black the right hand side.

The number of modes supported by the fiber is given by the number of intersections of the blue and red lines with the black line. The case shown has 2 modes supported. The condition for a single mode is that the x-intercept of the black line \( \sqrt{V^2 - k_x^2 a^2} \) occur before the first appearance of the first blue mode. This occurs at \( k_x a = \pi/2 \), so the condition for a single mode for this 1-D fiber is \( V < \pi/2 \).

By a similar reasoning, the total number of modes supported for this 1-D model is \( \text{Floor}(2V/\pi) + 1 \).

We can estimate the total number of modes supported by a square fiber by noting that for this case we require \( k_x^2 a^2 + k_y^2 a^2 + \kappa^2 a^2 = V^2 \). The highest spatial frequency modes have \( \kappa = 0 \), so this describes the first quadrant of a circle of radius \( V \). The number of modes is therefore approximately

\[
N = \frac{1}{4} \pi \left( \frac{2V}{\pi} \right)^2 = \frac{V^2}{\pi}
\]

which is close to the estimate \( V^2/4 \) we got from the last lecture.
Let us now consider the requirements for a single mode cylindrical fiber. The wave equation becomes

\[ \frac{d^2 E(r)}{dr^2} \frac{1}{r} \frac{dE}{dr} + \left( k^2(r) - k_z^2 \right) E(x) = 0 \]  

(8)

Defining \( k_r = \sqrt{k_1^2 - k_z^2} \), the cylindrically symmetric solutions to this are

\[ E_1 = J_0(k_r r) \quad r < a \]
\[ E_2 = K_0(k_r r) \quad r > a \]  

(9)

We could proceed as before to find the resonance frequencies by applying the boundary conditions. But we can figure out the requirements for a single mode by noting that the first excited mode must have a zero crossing at \( r < a \). Thus a single mode will be supported as long as \( k_r a \) is bigger than the first zero of \( J_0(k_r r) \) which occurs at \( k_r r = 2.405 \). Since \( k_r \) can be at most \( V/a \), this gives

\[ k_r a < V < 2.405. \]  

(10)

as the requirement for single mode operation of a fiber.