Lecture Notes, Relativity
Physics 448, Prof. Franz Himpsel

Concepts, Lorentz Transformation 2
Length Contraction, Time Dilation 3
Doppler Effect, Clocks, GPS, Redshift 5
Energy, Momentum, Particle Collisions 7
Four-Dimensional Space-Time, Tensors 8
Electromagnetism in 4D 10
From Relativity to Quantum Physics 11
Length Scales 12
Relativity

Concept:
• Coordinate systems moving with constant velocity are equivalent.
  No change in the equations of motion F=dp/dt, Maxwell's equations, the wave equation for light.
• The speed of light c is the same in all of them.

Consequences:
• Only relative motion is significant, there is no absolute reference frame, such as an ether (Michelson-Morley experiment).
• The velocity of light c is the highest velocity for transmitting energy and information.

General relativity:
• All coordinate systems are equivalent, even accelerated ones. Acceleration has the same effect as gravity (falling elevator example). Both cause space-time curvature and a slow-down of time.

Coordinate Transformations:

Define:
\[
\beta = \frac{v}{c} \quad (0 \ldots 1)
\]
\[
\gamma = \frac{1}{\sqrt{1-\beta^2}} \quad (1 \ldots \infty)
\]

Lorentz Transformation:
\[
x' = \gamma \cdot (x - \beta ct)
\]
\[
ct' = \gamma \cdot (ct - \beta x)
\]

Light beam:
\[
x = ct \quad x \quad ct \quad \Rightarrow \quad x' = ct'
\]
Speed of light is the same.

Length Contraction:
\[
L = \frac{\delta x}{\gamma} \leq L_0 \text{ (at rest)}
\]
Moving objects appear shorter.

Time Dilation:
\[
T' = \frac{\delta t}{\gamma} \geq T_0 \text{ (at rest)}
\]
Moving clocks appear to run slower.
• Measure length at the same time. For obtaining **length contraction** set $\delta t = 0$ in the Lorentz transformation $\delta x' = f(\delta x, \delta t)$.

Measure time at the same position. For obtaining **time dilation** set $\delta x = 0$ in the Lorentz transformation $\delta t' = f(\delta x, \delta t)$.

• Length is contracted and time dilated because they are derived in **different coordinate systems** ($\delta x$ versus $\delta t'$ above). The system $(x,t)$ describes an observer at rest for length contraction, and a moving observer for time dilation (with the clock at rest). Compare the muon decay below.

• The largest length and shortest time are measured at rest: “**Proper**” length and time $L_0$ and $T_0$.

• Contraction and dilation are **mutual** for two observers because of the **relativity principle**. In the twin paradox, both twins think the other is shorter and ages more slowly.

• Lengths contract along the motion ($x$), not perpendicular to it ($y$). Time always dilates.

Reason: $t$ is always involved in a Lorentz transformation, $y$ is not.

Examples:

• **Muon decay**: The *decay time* $\delta t'$ of a fast muon (or other particle) is longer than the “proper” decay time $T_0 = \tau$, measured with the muon at rest. This can be viewed *either* as time dilation delaying the decay of the muons, *or* as length contraction shortening the path they have to travel. The observer is at rest in the first view, and moving in the second view.

• **Twin paradox**: Two twins are moving with respect to each other and each twin thinks the other is aging more slowly due to time dilation. In order to compare their age properly, however, they have to meet in the same coordinate system before and after the trip, that is here on Earth. The twin on Earth never left this coordinate system and, therefore, can use **special relativity** to conclude that the other is younger. The twin in space experienced an acceleration during takeoff, turnaround, and landing. He has to use **general relativity**. See Physics FAQ: [http://math.ucr.edu/home/baez/physics/index.html](http://math.ucr.edu/home/baez/physics/index.html) (Special Relativity; The Twin Paradox)

Coping with Reference Frames:

• In relativity the same object can be viewed in several ways, which makes it tricky to set up a problem correctly. Usually, there is an **observer** and an **object**, which define two coordinate systems $K$ and $K'$. To avoid confusion it helps to use a **prime** for the frame where the **object** of interest is **moving**. However the math may become more cumbersome (for example when deriving time dilation).

• Use back-reflection of **light beams** for all measurements, since $c$ is constant.
Moving Coordinate Systems (=Frames):
The coordinate system \((x',t')\) moves with velocity \(v\) relative to \((x,t)\). The origin of the primed system \((x' = 0)\) corresponds to \(x = vt\).

\(x,y\) – Diagram: \(x,t\) – Diagram:

\[x = ct, \ x' = ct'\] (light line invariant)

Spacetime \((x,y,t)\):
The concept of simultaneous events depends on the frame. The causal relation of events remains invariant: Present, past, future, and unrelated events. They are separated by the invariant light cone.

Clock: Use bouncing light beam as oscillator.

moving clock: clock at rest:

\[\delta x = c \delta t\]

The light beam travels longer when the clock moves (time dilation).

Ruler: Clock + Light Beam. Use \(\delta x = c \delta t\).
Practical Clocks:

Instead of time and length, time and velocity are used as standards (atomic clock, c).
The second is defined as 9 192 631 770 microwave oscillations of a cesium spin-flip transition.
Defining $c = 299 792 458 \text{ m/s}$ fixes the meter in terms of the second.

The most precise clocks today are cesium fountain clocks with an accuracy of $10^{-15}$. A precision of $10^{-18}$ is expected within 3 years by going from microwaves to optical frequencies. That would be less than a second over the age of the universe. Even now, time can be measured 1000 times more accurately than any other quantity. An atomic clock contains two basic components, an oscillator that is locked onto an atomic frequency, and a counter.


Practical Ruler: The Global Positioning System (GPS)

Radio Beam + Atomic Clock  (in 24 Satellites). $c \approx= \text{1 foot / nsec}$

An absolute timing accuracy of 100 nsec defines the distance from satellite to user to 30 m. The differential mode achieves an accuracy below a meter by correcting timing errors. One needs distances from 3 satellites to define a point $x,y,z$ in space (2 intersecting spheres give a circle, a third sphere two points, one of which is usually off the Earth’s surface). A fourth satellite provides an accurate time $t$, thereby avoiding to carry an atomic clock in the receiver.


Doppler Effect

**Longitudinal:**  $v = v_0 \cdot [(1+\beta) / (1-\beta)]^{1/2}$  $\beta>0$ approaching, $\beta<0$ receding

(The light source moves parallel the line of sight.)

**Time dilation** and **source movement**.

**Transverse:**  $v = v_0 / \gamma$

(The light source moves perpendicular to the line of sight)

**Time-dilation** only.
Applications of the Doppler effect:

- Determination of the age of the universe from the redshift of receding galaxies.
- Cosmic background radiation is red-shifted ultraviolet \( \hbar \nu \approx 1 \text{ Rydberg} \).
- Laser cooling of atoms by recoil from Doppler-shifted laser light (clocks, lowest T).
- Mössbauer spectroscopy of ultra-sharp nuclear levels by tuning the velocity of a moving detector.
- Synchrotron radiation is a Doppler-shifted radio wave. The radiated power is \( \sim \gamma^4/r^2 \) and limits the energy of \( e^- e^+ \) colliders.

Laser Cooling, “Optical Molasses”:

An atom at rest (or receding from the laser) is not in resonance and does not absorb the laser light. If the atom approaches the laser, the laser frequency increases by the Doppler shift and coincides with the atom’s resonance frequency \( \nu_0 \). Absorption of a laser photon causes a recoil momentum \( p = \hbar / \lambda \) (De Broglie), which slows the atom down. The velocity is reduced gradually by reducing \( \delta \nu \).

In order to slow atoms moving in all directions, a set of 6 lasers is used (pointing along \( \pm x, \pm y, \pm z \)). This method reaches \( \mu K \) temperatures, which are used in atomic clocks. The temperature is reduced further into the \( nK \) regime by evaporative cooling, where the hottest atoms are released from a trap. Such temperatures are necessary to achieve Bose-Einstein condensation, where a group of atoms becomes so cold that they are all in the same ground state.

Superluminal Apparent Velocities in Quasars:

Matter falling into a black hole at the center of a galaxy produces narrow jets. There are several cases where clumps in these jets appear to move laterally with velocities greater than \( c \), e.g. 35 light years in 4 years of observation time for the radio source 3C 273. However, if the motion of these objects towards the observer is taken into account, their actual velocity does not exceed \( c \). Light emitted at later times has a shorter distance to travel.
Energy and Momentum

\[ \frac{(E/c)^2 - p^2}{(m_0c)^2} \]

Photon (\(m_0=0\)): \(E = pc\)

Relativistic Invariant:

Relativistic Energy:

\[ E = \gamma m_0 c^2 \]

Kinetic Energy:

\[ K = E - m_0 c^2 \]

Relativistic Momentum:

\[ p = \gamma m_0 v \]

Force:

\[ F = dp/dt \]

Particle Collisions

In units with \(c=1\)

Use the total energy-momentum:

\[ E_{\text{tot}} = \sum E_i \quad P_{\text{tot}} = \sum p_i \]

(\(\sum\) over all particles \(i\))

Total energy-momentum conservation:

\[ E'_{\text{tot}} = E_{\text{tot}} \quad P'_{\text{tot}} = P_{\text{tot}} \]

\((E_{\text{tot}}, P_{\text{tot}} \text{ and } E'_{\text{tot}}, P'_{\text{tot}} \text{ before/after reaction})\)

Lorentz invariance of \(E_{\text{tot}}^2 - P_{\text{tot}}^2\):

\[ E'_{\text{tot}}^2 - P'_{\text{tot}}^2 = E_{\text{tot}}^2 - P_{\text{tot}}^2 \]

\((E_{\text{tot}}, P_{\text{tot}} \text{ and } E'_{\text{tot}}, P'_{\text{tot}} \text{ in different frames})\)

Center-of-mass frame:

\(=\) zero total momentum, simplifies the algebra

\[ P_{\text{tot}} = \sum p_i = 0 \]

Convert individual momenta \(p_i\) to energies:

\(\text{(To reduce the number of variables)}\)

Approximations \(E \gg m, E \approx m\):

\(E \gg m:\)

\[ p_i \approx E_i - m_i^2/2E_i \]

\(E \approx m:\)

\[ p_i \approx \sqrt{2m_i(E_i-m_i)} \]

Energy \(E_i'\) per particle in a collider, compared to
the energy \(E_i\) of a particle hitting a fixed target:

\[ E_i' = \sqrt{\frac{1}{2}(E_i+m_0)m_0} \]

\(\rightarrow \sqrt{\frac{1}{2}E_i m_0}\) for \(E_i \gg m_0\)
Four-Dimensional Space-Time

Greek indices run from 0 to 3. Automatic summation over equal upper and lower indices.

Vectors: \[ x^\mu = (ct, x, y, z) = (ct, x) \]
(1 index) \[ x_\mu = (ct, -x, -y, -z) = (ct, -x) \]
Energy-Momentum \[ p^\mu = \left( \frac{E}{c}, p_x, p_y, p_z \right) = \left( \frac{E}{c}, p \right) \]
\[ p_\mu = \left( \frac{E}{c}, -p_x, -p_y, -p_z \right) = \left( \frac{E}{c}, -p \right) \]
Derivative \[ \partial_\mu = \partial/\partial x^\mu \]
Scalar Product \[ x \cdot y \]

Scalars: \[ x_\mu x^\mu : \Delta x_\mu \Delta x^\mu = \Delta(x^2) - (\Delta ct)^2 = \Delta s^2 \]
Space-Time Interval
- \(< 0\) time-like, correlated
- \(> 0\) space-like, uncorrelated
- \(= 0\) light-like

Rest Mass \[ p_\mu p^\mu : p_\mu p^\mu = \left( \frac{E}{c} \right)^2 - p^2 = (m_0c)^2 \]

Plane Wave \[ e^{i \hbar p_\mu x^\mu} = e^{i \hbar (px-ct)} = \psi(x^\mu) \]

Metrics:
( Number of indices = rank) \[ g_{\mu \nu} = g^{\mu \nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \]

Metric Tensor
- raises/lowers indices:
  - \( x^\mu = g^{\mu \nu} x_\nu \)
  - \( x_\mu = g_{\mu \nu} x^\nu \)

Spherical Symmetry \[ x^\mu = (ct, r, \vartheta, \varphi) \]
- general relativity: black holes, models of the universe

Antisymmetric Tensor \[ -\varepsilon_{\mu \nu \rho \sigma} = \varepsilon^{\mu \nu \rho \sigma} = +1 \text{ if } \mu \nu \rho \sigma \text{ is an even (odd) permutation of } 0123 \]
- otherwise

(Generates pseudotensors and dual tensors)
Coordinate Transformations

\[ x'^\mu = \Gamma^\mu_\nu x^\nu \quad \text{Linear Transformation} \]

\[ x'_\mu y'^\mu = x_\mu y^\mu \quad \text{Scalar Product is Invariant} \]

\[ \det(\Gamma) = +1 : \]

Lorentz Transformation \( L_\gamma \):

\[
\Gamma^\mu_\nu = \begin{pmatrix}
\gamma & -\beta \gamma & 0 & 0 \\
-\beta \gamma & \gamma & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

\[ \beta, \gamma \text{ see p. 2} \]

In the \((ct,x)\) Plane

\[ \theta = \text{Rapidity} \]

Rotation \( R_\phi \):

\[
\Gamma^\mu_\nu = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \phi & \sin \phi & 0 \\
0 & -\sin \phi & \cos \phi & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

In the \((x,y)\) Plane

\[ \phi = \text{Rotation Angle} \]

\[ \det(\Gamma) = -1 : \]

Time Reversal \( T \):

\[ t' = -t \quad x' = +x \]

Space Inversion \( P \):

\[ t' = +t \quad x' = -x \quad \text{Determines parity.} \]

Separates tensors from pseudo-tensors.

General Lorentz Group:

Proper Lorentz Group

- L, R
- T
- P
- PT
Electromagnetism in Four Dimensions

Current Density: \( J^\mu = (c\rho, j^x, j^y, j^z) = (c\rho, j) \) charge/current density

Potentials: \( A^\mu = (\varphi, A^x, A^y, A^z) = (\varphi, A) \) Coulomb/vector potential

Field Tensor: \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \)
\[
\begin{pmatrix}
0 & E_x & E_y & E_z \\
-E_x & 0 & -B_z & B_y \\
-E_y & B_z & 0 & -B_x \\
-E_z & -B_y & B_x & 0
\end{pmatrix}
\]

\( E, B = (E_x, E_y, E_z), (B_x, B_y, B_z) \)

\( F_{\mu\nu} \leftrightarrow F^{\mu\nu} : \)

\( E \Rightarrow -E \)
\( B \Rightarrow B \)

Dual Field Tensor: \( \tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} \)
\[
\begin{pmatrix}
0 & B_x & B_y & B_z \\
-B_x & 0 & E_z & -E_y \\
-B_y & -E_z & 0 & E_x \\
-B_z & E_y & -E_x & 0
\end{pmatrix}
\]

Maxwell's Equations:
\( \partial_\mu F^{\mu\nu} = 4\pi/c \cdot J^\nu \) inhomogeneous
\( \partial_\mu \tilde{F}^{\mu\nu} = 0 \) homogeneous

Continuity Equation: \( \partial_\nu J^\nu = 0 \) charge conservation

Equation for Potentials: \( \partial_\nu \partial^\nu A^\mu = 4\pi/c \cdot J^\mu \) with extra condition \( \partial_\mu A^\mu = 0 \)

Wave Equation: \( \partial_\nu \partial^\nu A^\mu = 0 \) in vacuum \( (J^\mu = 0) \)

Energy Momentum (Density) Tensor: \( T^{\mu\nu} \)
\[ T^{00} = \frac{(E^2 + B^2)}{8\pi} \]
energy density of the EM field

General Relativity: \( 8\pi G/c^4 \ T^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R^\lambda_\lambda \)
\( T^{\mu\nu} \) determines the space-time curvature tensor \( R \), \( G = \) gravitational constant

(Factors \( 4\pi\varepsilon_0 \) etc. appear in other unit systems.)
From Relativity to Quantum Physics

Quantum physics describes the simultaneous occurrence of particle and wave properties by using a wave function $\psi$, whose square $|\psi|^2$ is the probability of finding a particle.

- **Particles** are characterized by their position $x^\mu$ and their momentum $p^\mu$:
  $$x^\mu = (ct, \mathbf{x}) \quad p^\mu = (E/c, \mathbf{p})$$

- **Waves** are characterized by their wave function, for example a plane wave:
  $$\psi(x^\mu) = \exp[-i/\hbar \cdot p_\mu x^\mu] = \exp[i/\hbar \cdot (\mathbf{p} \cdot \mathbf{x} - Et)]$$

Energy and momentum $p_\mu$ of the particle are obtained by differentiating $\psi(x^\mu)$ with respect to $x^\mu$:

$$i\hbar \partial_\mu \psi = p_\mu \psi \quad i\hbar \partial/\partial t \psi = E \psi \quad -i\hbar \partial/\partial x \psi = p \psi$$

A generalization suggests **substituting numbers (classical) by operators (quantum)**, which act on a wave function $\psi(x^\mu)$:

$$p^\mu \Rightarrow i\hbar \partial^\mu \quad E \Rightarrow i\hbar \partial/\partial t \quad p \Rightarrow -i\hbar \partial/\partial x \quad \partial^\mu = \partial/\partial x_\mu = (\partial/\partial ct, -\partial/\partial \mathbf{x})$$

### Relativistic Wave Equations

**Spin 0**, Scalar $\psi$:

$$p_\mu p^\mu = (m_0 c)^2$$

$$\downarrow \quad \downarrow$$

$$-\hbar^2 \partial_\mu \partial^\mu \psi = (m_0 c)^2 \cdot \psi$$

**Klein-Gordon equation**

**Spin 1**, Vector $A_\nu$:

$$\partial_\mu \partial^\mu A_\nu = 0$$

**Photon**

**Wave equation**, $m_0=0$, extra “gauge” condition $\partial^\nu A_\nu = 0$

**Spin 2**, Tensor $g_{\mu\nu}$:

**Graviton**

**Spin 1/2**, Spinor $\psi$:

$$i\hbar \gamma^\mu \partial_\mu \psi = m_0 c \cdot \psi$$

**Electron**

**Dirac equation**, $\gamma^\mu = $ Dirac matrices, $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$

(See the quantum physics notes, p. 17)
**Length Scales**

Use fundamental constants to define fundamental lengths:

c = velocity of light (relativity), \( \hbar = \) Planck’s constant (quantum mechanics), \( G = \) gravitational constant (general relativity), \( e = \) electron charge (electromagnetism), \( m_e = \) electron mass.

Three units define all physical quantities, e.g. \( \hbar, c, G \) (cosmology) or \( \hbar, e, m_e \) (atomic).

(Using \( h \) instead of \( \hbar \) generates a slightly different set of lengths).

Several of these lengths are connected by the dimensionless fine structure constant:

\[
\alpha = \frac{e^2}{\hbar c} \approx 1 / 137.036
\]

**Hubble Radius** \( R_H = 1.4 \cdot 10^{26} \text{ m} \)

Radius of the observable universe \( \approx c \cdot \) age of the universe; longest physical length.

**Bohr Radius** \( a_0 = \frac{\hbar^2}{m_e e^2} = \frac{\lambda_C}{\alpha} = 5.3 \cdot 10^{-11} \text{ m} \)

Radius of the hydrogen atom.

**Compton Wavelength / 2\pi** \( \lambda_C = \frac{\hbar}{m_e c} = 3.9 \cdot 10^{-13} \text{ m} \)

Wavelength corresponding to the rest energy \( m_e c^2 \) of the electron; vacuum becomes populated by short-lived \( e^- e^+ \) pairs; regime of quantum electrodynamics.

**Classical Electron Radius** \( r_0 = \frac{e^2}{m_e c^2} = \lambda_C \cdot \alpha = 2.8 \cdot 10^{-15} \text{ m} \)

The self-energy of the electron’s classical Coulomb field equals \( m_e c^2 \). Irrelevant, because quantum electrodynamics takes over below \( \lambda_C \) and reduces the electron self-energy.

**Planck Length** \( l_P = (G\hbar/c^3)^{1/2} = 1.6 \cdot 10^{-35} \text{ m} \)

Space-time becomes quantized; regime of quantum gravity; shortest physical length.