In a superconductor, electrons form pairs which are bound by an energy $E_g$ (the superconducting gap). The pairs are held together by exchanging phonons (lattice vibrations). As analog, an electron bound to a positron ($e^+$) is shown below, where the Coulomb attraction can be described by the exchange of photons. More generally, forces are created by exchanging Bosons. Pairs in high temperature superconductors are held together by exotic Bosons that have yet to be discovered. In real space, the presence of phonons may be viewed as positive ions moving into the space between the negative electrons and mediating an attraction between the electrons.

### Visualizing pairs (difficult!)

The picture becomes more complicated by the fact that there are $10^7$ other pairs within the diameter $\xi_p$ of a pair, all of which contribute to the electron-electron attraction. A typical value of $\xi_p$ can be estimated from the Fermi velocity $v_F \approx 10^6\text{m/s}$ and the vibration period of a phonon $1/\nu \approx 10^{-13}\text{s}$: $\xi_p \approx v_F/\nu \approx 0.1\mu\text{m}$.

Basically, the positive ions take a long time to get going, and by that time the electron has already sped away by a distance $\xi_p$.

**Evidence for pairs:** The magnetic flux $\Phi$ is quantized in units of $2e$: $\Phi_0 = h/q = h/2e$

**Evidence for phonons:** The critical temperature $T_c$ is proportional to $1/\sqrt{m}$ for different isotopes of an element ($m =$ atomic mass), and so is the phonon frequency.
Quantum numbers of pairs

<table>
<thead>
<tr>
<th>Quantum number</th>
<th>$\varepsilon_1$</th>
<th>$\varepsilon_2$</th>
<th>pair</th>
</tr>
</thead>
<tbody>
<tr>
<td>Momentum: $p=\hbar k$</td>
<td>+$k$</td>
<td>-$k$</td>
<td>0</td>
</tr>
<tr>
<td>Spin: $s$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
</tr>
<tr>
<td>$m_s$</td>
<td>+$\frac{1}{2}$</td>
<td>-$\frac{1}{2}$</td>
<td>0</td>
</tr>
<tr>
<td>Orbital angular momentum: $l$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

“singlet” pairing
“s-wave” pairing ($l=2$ d-wave in HItC)

Why is the resistance zero in a superconductor?
Electrical resistance is caused by electrons losing energy by scattering. The electron pairs are all in their ground state and cannot lose energy. When accelerated to carry a current the pairs acquire kinetic energy, but they still cannot lose energy as long as their kinetic energy is smaller than the energy $E_g$ to break up a pair.

Is the resistance really zero?
Experimental answer: A lower limit of $10^5$ years has been established for the decay time $\tau$ of the current $I$ in a superconducting loop ($10^{-5}$ precision over a year). $I = \exp(-t/\tau)$, $\tau = L/R$, $L$ = inductance, $\Rightarrow$ resistance $R$.
Theoretical answer: The probability for a superconductor becoming resistive is $\ll \exp(-10^7)$. This number represents the Boltzmann factor $\exp(-10^7E_g/k_B T)$ for the probability of breaking the $\approx 10^7$ pairs within a coherence length $\xi$ at a temperature $T \ll T_c$ ($E_g \approx 3.5k_BT_c$). If only one of the pairs remained, it would carry the current with zero resistance and pass it on to another pair in the adjacent coherence volume.

What kills superconductivity?
Anything that provides an energy $E \geq E_g(T)$ to each electron pair.
1. Thermal energy: Critical temperature $T_c$
2. Kinetic energy: Critical current $j_c$
3. Magnetic energy: Critical field $H_c$
Results from BCS theory

The BCS theory gives a microscopic explanation of the many-electron interaction that binds the electrons into pairs. It makes predictions of the key parameters, such as the critical temperature $T_c$ and the value of the gap $E_g = 2\Delta$. They are determined by a characteristic phonon energy $\hbar\omega_{ph}$, the density of states at the Fermi level $D(E_F)$, and the electron-phonon interaction energy $V$:

$$E_g \approx 3.5 \cdot k_B T_c$$

$$k_B T_c \approx \hbar \omega_{ph} \cdot e^{-\frac{1}{D(E_F)V}}$$

The maximum critical temperature scales with the phonon frequency $\omega_{ph} \approx \sqrt{\kappa/m}$ ($\kappa$ = force constant, $m$ = mass). The $1/\sqrt{m}$ dependence produces an isotope effect, which shows that phonons are involved in superconductivity. The density of states at the Fermi level $D(E_F)$ plays a critical role in the exponent (compare the Stoner criterion for ferromagnetism). Only electrons close to the Fermi level $E_F$ are able to lower their energy when the superconducting gap opens up (lightly hatched area). Each electron gains about $\Delta$, i.e., a pair gains about $E_g$.

**Measuring the gap $E_g = 2\Delta$**

1. **Tunneling** (metal to superconductor): The derivative $dI/dV$ of a current-voltage $I(V)$ measurement is approximately proportional to the density of states $D(E)$. $I=0$ for $|V|<\Delta/e$.
2. **Infrared Absorption**: The absorption coefficient is measured versus the frequency of infrared or microwaves. Photons cannot be absorbed by a superconductor for $h\nu < E_g$. 
3. **Photoemission**: There are no photoelectrons emitted with initial energies between \( E_F - \Delta \) and \( E_F \) in the superconducting state, whereas the spectrum extends to all the way up to \( E_F \) in the normal state. Photoemission detects the \( k \)-dependence of \( E_g \).

In all these experiments, the **normal state** just above \( T_c \) is compared to the **superconducting state** below \( T_c \) to eliminate all effects other than superconductivity.

**Length scales**

1. The **penetration depth** \( \lambda \) is the decay constant of the **magnetic field**.
2. The **coherence length** \( \xi \) is the decay constant of the **pair density**.

\[
 B(z) \sim \exp(-z/\lambda) \\
 n(z) \sim [1 - \exp(-z/\xi)]
\]

**Type I versus Type II superconductors**

A) **Type I**: \( \xi \gg \lambda \), **Type II**: \( \xi << \lambda \)

B) **Type I** superconductors have a **single** (rather low) **critical field** \( H_c \), **Type II** superconductors have **two critical fields** \( H_{c1} \) and \( H_{c2} \). Between \( H_{c1} \) and \( H_{c2} \) the magnetic field penetrates part of the superconductor. The magnetic flux breaks up into individual flux quanta \( \Phi_0 = \frac{\hbar}{2e} = 2 \cdot 10^{-15} \text{ Tm}^2 \) (= vortices). The show up as bright spots in the figure, where states at \( E_F \) (inside the gap) are measured by scanning tunneling microscopy.

C) **Type I** superconductors are **pure** materials, **Type II** superconductors are **alloys** with **pinning centers** for the **magnetic flux**.
High Temperature Superconductors (HiTc)

Superconductivity occurs in CuO planes, which are stabilized by an ionic lattice.

The superconducting carriers are holes introduced by doping.

The big open question is the nature of the Boson that gives rise to pairing (see p. 1).