Homework 3 Solutions

W1) We are given
\[ f(x) = \begin{cases} \frac{1}{2}x^2 & \text{if } 0 \leq x \leq x_0 \\ 0 & \text{elsewhere} \end{cases} \]

(a) I will treat \( f(x) \) as a probability distribution, so I want to normalize as follows:

\[ \int_0^{x_0} f(x) \, dx = 1 \Rightarrow \frac{1}{2} \int_0^{x_0} x^2 \, dx = A \]

\[ A = \frac{2}{x_0^2} \]

(b) \( \bar{x} = N \int_0^{x_0} x f(x) \, dx \). Using \( f(x) = \frac{1}{N} n(x) \) we have:

\[ \bar{x} = \int_0^{x_0} x f(x) \, dx = A \int_0^{x_0} x^2 \, dx = A \frac{x_0^3}{3} = \left( \frac{2}{x_0^2} \right) \left( \frac{x_0^3}{3} \right) \Rightarrow \bar{x} = \frac{2}{3} x_0 \]

(c) To get \( X_{\text{rms}} \) we first need \( \bar{x}^2 \):

\[ \bar{x}^2 = \int_0^{x_0} x^2 f(x) \, dx = A \int_0^{x_0} x^4 \, dx = A \frac{x_0^5}{5} = \left( \frac{2}{x_0^2} \right) \left( \frac{x_0^5}{5} \right) = \frac{x_0^3}{2} \]

so

\[ X_{\text{rms}} = \sqrt{\bar{x}^2} \Rightarrow X_{\text{rms}} = \frac{1}{\sqrt{2}} x_0 \]

W2) (a) To find the fraction of atoms in each state we use the Maxwell-Boltzmann formula. Combining equations (10.4) and (10.3)

\[ n_i = B g_i e^{-E_i/kT} \]

where \( B \) is a normalization constant and where the degeneracy factors \( g_i \) are all \( g_i = 1 \).

For \( T = 300 \text{K} \) we get:

\[ kT = (8.617 \times 10^{-5} \text{eV/K}) \times 300 \text{K} = 2.58 \times 10^{-2} \text{eV} = 0.0259 \text{eV} \]

If we have \( N \) atoms, then the constant \( B \) is determined as follows:

\[ E_3 = 0.2 \text{eV} \]

\[ E_2 = 0.1 \text{eV} \]

\[ E_1 = 0 \]
\[ N = n_1 + n_2 + n_3 = B \left[ e^{-E_1/kT} + e^{-E_2/kT} + e^{-E_3/kT} \right] = B \left[ 1 + 0.0004 + 0.0004 \right] \]
\[ = 1.001 \ B \quad \Rightarrow \quad B = N/1.0013 \]

The fraction of atoms in state \( i \) is

\[ f_i = n_i/N = B g_i e^{-E_i/kT} / N = \left( \frac{1}{1.0013} \right) e^{-E_i/kT} \]

Thus

\[ f_1 = \left( \frac{1}{1.0013} \right) e^0 \quad \Rightarrow \quad f_1 = 0.979 = 97.9 \% \]

\[ f_2 = \left( \frac{1}{1.0013} \right) e^{-0.1/0.0859} \quad \Rightarrow \quad f_2 = 0.0205 = 2.05 \% \]

\[ f_3 = \left( \frac{1}{1.0013} \right) e^{-0.2/0.0859} \quad \Rightarrow \quad f_3 = 0.00043 = 0.043 \% \]

(b) \[ \bar{E} = \frac{1}{N} \sum_i E_i n_i = \sum_i E_i f_i \]

\[ = (0.979)(0) + (0.0205)(0.1 \text{eV}) + (0.00043)(0.2 \text{eV}) \quad \Rightarrow \quad \bar{E} = 2.13 \times 10^{-3} \text{eV} \]

Notice that in this example \( \bar{E} \) is small compared to \( kT \). Even if the atom had more states at higher energies, \( \bar{E} \) would not be affected much since these states would have very small \( f_i \)’s.

10-6 (a) \[ n_1 + n_2 = 10^{20} \]

\[ n_2 = \exp \left( \frac{-4.86 \text{eV} \times 1.602 \times 10^{-19}}{1.38 \times 10^{-23} \text{J/K} \times 1.600 \times 10^3 \text{K}} \right) = 4.98 \times 10^{-16} \]

Assuming \( n_1 = 10^{20} \),

\[ n_2 = n_1 \left( 4.98 \times 10^{-16} \right) = \left( 10^{20} \right) \left( 4.98 \times 10^{-16} \right) = 4.98 \times 10^4 \]

(b) Power emitted = number of photons emitted/s x (energy/photon)

\[ = \left( \frac{1}{T} \right) x n_2 \times 4.86 \text{eV} \]
\[ = 10^7 \text{s}^{-1} \times 4.88 \times 10^4 \times 4.86 \text{eV} \times 1.602 \times 10^{-19} \text{J/eV} \]
\[ = 3.88 \times 10^{-7} \text{J/s} \]
\[ = 0.388 \text{μW} \]
10-8 (a) Replacing $E$ with $K$, the kinetic energy, we have $n(K)dK = \frac{2\pi(N/V)}{(\pi k_B T)^{3/2}} K^{1/2} e^{-K/k_B T} dK$.

The most probable value of kinetic energy, $K_{mp}$, occurs where $K^{1/2} e^{-K/k_B T}$ has a maximum:

$$\frac{d}{dK} \left[ \frac{1}{2} K^{1/2} e^{-K/k_B T} + K^{1/2} e^{-K/k_B T} \left( -\frac{1}{k_B T} \right) \right] = 0$$

implies $K_{mp} = \frac{k_B T}{2}$.

(b) $\bar{K} = \frac{1}{N/V} \int_{K=0} K n(K) dK = \int_{K=0} \frac{2\pi}{(\pi k_B T)^{3/2}} K^{3/2} e^{-K/k_B T} dK = \frac{2\pi}{(\pi k_B T)^{3/2}} \left[ \frac{6(\pi k_B T)^{1/2}}{(8(1/k_B T)^2)} \right]$

Note that our result $\bar{K} = \frac{3}{2} k_B T$ agrees with the equipartition theorem.

(c) $K_{ms} = \left[ \frac{1}{N/V} \int_{0} K^{2} n(K) dK \right]^{1/2} = \left[ \frac{2\pi}{(\pi k_B T)^{3/2}} \int_{0} K^{5/2} n(K) dK \right]^{1/2}$

$$= \left[ \frac{2\pi}{(\pi k_B T)^{3/2}} \left( \frac{15}{16} (1/k_B T)^3 \right)^{1/2} \right] = \left[ \frac{15}{4} (k_B T)^2 \right]^{1/2}$$

or $K_{ms} = \left( \frac{15}{4} \right)^{1/2} k_B T$.

3-2 Assume that your skin can be considered a blackbody. One can then use Wien’s displacement law, $\lambda_{max} = 0.289 \times 10^{-2} \text{ m} \cdot \text{K}$ with $T = 35^\circ \text{C} = 308 \text{ K}$ to find

$$\lambda_{max} = \frac{0.289 \times 10^{-2}}{308 \text{ K}} = 9.41 \times 10^{-6} \text{ m} = 9.41 \text{ nm}.$$  

3-4 (a) From Stefan’s law, one has $\frac{P}{A} = \sigma T^4$. Therefore,

$$\frac{P}{A} = (5.7 \times 10^{-8} \text{ W/m}^2 \cdot \text{K}^4)(3000 \text{ K})^4 = 4.62 \times 10^6 \text{ W/m}^2.$$

(b) $A = \frac{P}{4.62 \times 10^6 \text{ W/m}^2} = \frac{75 \text{ W}}{4.62 \times 10^6 \text{ W/m}^2} = 16.2 \text{ mm}^2$. 
(3) (a) We need to know the surface area of the sun. For a sphere

\[ A = 4\pi R^2 = 4\pi (7 \times 10^9 \text{ m}) = 6.16 \times 10^{18} \text{ m}^2 \]

The total radiated power will be

\[ P = (5 \cdot T^4) \cdot A = (5.67 \times 10^{-8} \text{ W/m}^2 \cdot \text{K}^4)(5800 \text{ K})^4 (6.16 \times 10^{18} \text{ m}^2) \]

\[ P = 3.59 \times 10^{26} \text{ W} \]

(b) We have \( P = \frac{dE}{dt} = 3.59 \times 10^{26} \text{ Joules/sec} \). If \( E = mc^2 \), then

\[ \frac{dE}{dt} = c^2 \frac{dm}{dt} \quad \Rightarrow \quad \frac{dm}{dt} = \frac{dE}{dt} / c^2 = (3.59 \times 10^{26} \text{ J/s}) / (3 \times 10^8 \text{ m/s})^2 \]

\[ \frac{dm}{dt} = 4.39 \times 10^4 \text{ kg/s} \]

(c) During the lifetime of the sun, we expect the mass to decrease by a total of about

\[ \Delta m = 0.01 \text{ m} = 2 \times 10^{28} \text{ kg} \]

So the lifetime is about

\[ \Delta t = \frac{\Delta m}{\frac{dm}{dt}} = \frac{(2 \times 10^{28} \text{ kg})}{(4.39 \times 10^4 \text{ kg/s})} = 4.56 \times 10^8 \text{ s} \]

\[ \Delta t = 4.56 \times 10^8 \text{ s} = 1.44 \times 10^{11} \text{ years} \]