Welcome to the String Theory Seminar Series

Tuesdays, 3:00pm, Chamberlin 5280

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Black holes in Gödel universes and plane waves

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We find the black hole solution in asymptotically plane-wave/Gödel universe space-time in supergravity.
Solutions of general theory of relativity are classified according to their asymptotics

- Flat
- Compact
- Freedman-Robertson-Walker
- De-Sitter
- Anti de-Sitter

Black holes in flat space: Schwarzschild solution

Rules of quantum field theory, i.e. scattering, are very different depending on the asymptotics
$AdS_5 \times S_5$ is an important asymptotic geometry

\[ ds^2 = \frac{R^2}{\cos^2 \rho} (-dt^2 + d\rho^2 + \sin^2 \rho d\Omega^2_3) + R^2 d\Omega^2_5 \]

\[ F_5 = N (d\Omega_5 + *d\Omega_5) \]

Very different from asymptotically flat space because it has a boundary
While still a conjecture, string theory in this background can be reformulated as gauge theory living on the boundary.

Extremely interesting because it relates the (unknown) quantum theory of gravity to the (familiar) theory of glons and gluinos.

Black holes in anti de-Sitter spaces corresponding to field theories at finite temperature.

Calculations on the string theory side was very very hard!
Plane waves are also important backgrounds in string theory.

There are space-times of the form

\[ ds^2 = -dx^+ dx^- + B_{ij}(x^+) x^i x^j (dx^+)^2 + (dx^i)^2 \]
**Penrose Limit**

Plane waves are ubiquitous in geometry

Take any null geodesics and take the near geodesic limit
This limit can be applied to null geodesics in $AdS_5 \times S_5$. 
This gives rise to a plane wave geometry with Background 5-form

\[ F_{+1234} = F_{+5678} = \text{const} \]

For this plane wave:
- The string theory in plane wave is much easier
- This limit has interpretation in gauge theory (BMN)

Lots of computations are doable.
- Source of great excitement!
Gödel Universes: 

\[ ds^2 = -(dt + j r^2 d\phi)^2 + dr^2 + r^2 d\phi^2 \]

Strange: closed time like curves through every point
Gauntlett et.al: Gödel Universes are supersymmetric solutions of supergravity *(September 2002)*

Boyda et.al: Gödel Universe are related to plane waves by T-duality/dimensional reduction *(December 2002)*

\[
-dt^2 + dy^2 - j^2 r^2 (dt + dy)^2 + dr^2 + r^2 d\phi^2 \\
\phi = \tilde{\phi} - j(t + y) \\
-(dt + jr^2 d\tilde{\phi})^2 + dr^2 + r^2 d\tilde{\phi}^2 + (dy^2 - jr^2 d\tilde{\phi})^2
\]

In string theory, planes wave and Gödel universes should be considered to be in the same class
Both plane waves and Gödel universes are non-hyperbolic.

That is to say the space do not admit a foliation in terms of everywhere space-like hypersurfaces

- Meaning of initial value problems are obscure
- Issue of causality and chronology protection is obscure
- Notion of Hilbert-space of functions obscure
- Notion of for fields living in these spaces obscure
We are concerned with solving the equation of motion of type IIA/B supergravity

\[
D^M P_M = \frac{1}{24}\kappa^2 G_{MNP} G^{MNP}
\]

\[
D^P G_{MNP} = P^P G^*_{MNP} - \frac{2}{3} i\kappa F_{MNOPQR} G^{PQR}
\]

\[
-R_{MN} = P_M P^*_N + P^*_M P_N + \frac{1}{6}\kappa^2 F_{MP_1...P_4} F^{P_1...P_4 N}
\]

\[
+\frac{1}{8}\kappa^2 (G^P_{MPQ} G^*_{NPQ} + G^*_{MPQ} G^P_{NPQ})
\]

\[
-\frac{1}{6} g_{MN} G^{PQR} G^*_{PQR}
\]

Eq. (13.1.27), page 320 of GSW

Why? They have something to do with string theory
Immediately following BMN many people asked about black holes in pp-wave backgrounds

- Black hole solution in AdS space taught us a lot about SYM at finite temperature

However, direct analysis did not go very far because the supergravity ansatz was too complicated.

- It has become a kind of an obsession: a goal with its own merit
In this talk, I would like to report on a breakthrough which lead to the discovery of exact solution of type II supergravity which describes a black hole in Gödel universes (black strings in plane waves)


Main Encouragement

Gauntlett et.al. broke the problem down to

1) find the Gödel solution in 5d minimal supergravity
2) embed the 5d solution into 11d supergravity

Enough to find Gödel black hole in 5d theory \((g_{\mu\nu}, A_\mu)\).

In fact, Gauntlett et.al. found all supersymmetric solutions

Herdeiro: one of these solutions was the extreme (zero temperature) black hole in 5d Gödel universe.
5d Godel universe:

\[ ds^2 = -(dt + jr^2\sigma)^2 + dr^2 + r^2d\Omega_3^2 \]

\[ A = \frac{\sqrt{3}}{2} jr^2\sigma \]

\[ \frac{r^2\sigma}{2} = (x^1dx^2 - x^2dx^1 + x^3dx^4 - x^4dx^3) \]

Try an ansatz

\[ ds^2 = -f(r)dt^2 - g(r)r\sigma_L^3 dt - h(r)r^2(\sigma_L^3)^2 \]

\[ + k(r)dr^2 + r^2d\Omega_3^2, \]

\[ A = \frac{\sqrt{3}}{2} jr^2\sigma_L^3. \]

Mimicking the form of rotating black hole in 5d
\[
\begin{align*}
\text{In[1]} &= \text{eq} \left\{ \begin{array}{l}
-128 g[x]^2 k[x]^2 - 32 j^2 f[x]^2 k[x] (-1 - (-4 h[x]) k[x]) + x^2 (-1 - 4 h[x]) k[x] f[x]^2 - 8 g[x] k[x] f[x] g[x] [x] + 4 g[x] f[x] (-f[x] k[x] k[x] - 2 k[x] f[x] + f[x] f[x]) + f[x] (-8 g[x] k[x] (-1 - (-4 h[x]) f[x]^2 - 4 h[x]) k[x]) + 16 g[x] k[x] g[x] ^2 + x (-4 h[x]) f[x] k[x] \cdot \\
\text{Out[1]} &= \text{h}\left( f[x] (6 - 24 h[x] - 4 h[x] f[x]) + 2 (4 g[x] [x] + (-1 - 4 h[x]) f[x] f[x]) / (4 z^2 (f[x] + 4 g[x]^2 - 4 f[x] h[x]) k[x] k[x]) \right), \quad \text{Out[2]} = \text{g}.
\end{array} \right.
\end{align*}
\]
It is remarkable that one finds

\[
f(r) = 1 - \frac{2m}{r^2}, \\
g(r) = 2jr, \\
h(r) = j^2 (r^2 + 2m), \\
k(r) = \left(1 - \frac{2m}{r^2} + \frac{16j^2m^2}{r^2}\right)^{-1}
\]

is a solution

5d Godel Black Hole (Black string in 6d pp wave)

For how we found this solution, please ask privately.
Instead, let me describe a general solution generating technique which reproduces this answer (and generalize a lot) 

(GHHLR)

We were led to seek such a technique because we wanted to find analogous solutions in higher dimensions.

In that case, the supergravity equations did not truncate to a simple \((g_{\mu\nu}, A_\mu)\) system.

and the equations were a mess

There had to be a better way
The better way: starting with Minkowski space, and

1. Compactify
2. T-duality
3. Twist: \( x^\mu \rightarrow \Lambda^\mu_\nu x^\nu \)
4. T-duality
5. Decompactify

is a solution generating technique for Melvin universe
T-duality

Consider compactification on $T^N$

Consider components of the metric $G_{ij}$ and the Kalb-Ramond field $B_{ij}$ polarized along $T^N$.

Assemble them into a $2N \times 2N$ matrix

$$M = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix}$$
T-duality:

If $M$ is a solution to the equation of motion, so is

$$M' = \Lambda M \Lambda^T, \quad \Lambda \in O(N, N, \mathbb{Z})$$

plus appropriate transformation for other fields (again, look up the details in books)

Nice because this manipulation is algebraic. It is much easier than solving PDE’s
Twist

Make the periodic identification

\[(y, \phi) = (y + L, \phi + \alpha L) .\]
Equivalently, make a change of variables

$$\tilde{\phi} = \phi - y$$

and identify

$$(y, \tilde{\phi}) = (y + L, \tilde{\phi}) .$$

• This adds new terms to the line element)
Melvin Universe

Space-time of the form

\[ ds^2 = -dt^2 + \frac{1}{1 + \alpha^2 r^2} dy^2 + \frac{\alpha^4 r^4}{4(1 + \alpha^2 r^2)} \sigma^2 + r^2 d\Omega_7^2 \]

\[ B = \frac{\alpha r^2}{2(1 + \alpha^2 r^2)} \sigma \wedge dy \]

\( \sigma \) is an one form

\[ r^2 \sigma = x^T \Lambda dx \]

This is a geometry is supported by the flux which is localied near \( r = 0 \).
Back to the better way: starting with Minkowski space, and

1. Compactify
2. T-duality
3. Twist: $x^\mu \rightarrow \Lambda^\mu_\nu x^\nu$
4. T-duality
5. Decompactify

is a solution generating technique for Melvin universe

6. Infinite boost
gives rise to plane wave (if the boost and twist is scaled)
This is like taking the Penrose limit of the Melvin universe
“T-dualizing and twisting along a null direction”

(Alishahiha, Ganor)

Do the same thing with a black string

0. un-Boost
1. Compactify
2. T-duality
3. Twist: $x^\mu \rightarrow \Lambda^\mu_{\nu} x^\nu$
4. T-duality
5. Decompactify
6. Boost
This turns out to give rise to a black string in plane waves!

Rank of \( \Lambda \) determines the dimension of the plane wave.

\[
\begin{align*}
    ds_{str}^2 &= -\frac{f(r) \left(1 + \beta^2 r^2\right)}{k(r)} dt^2 - \frac{2 \beta^2 r^2 f(r)}{k(r)} dt dy \\
    &+ \left(1 - \frac{\beta^2 r^2}{k(r)}\right) dy^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_7^2 - \frac{\beta^2 r^4 \left(1 - f(r)\right)}{4 k(r)} \sigma^2
\end{align*}
\]

\[
\begin{align*}
    f(r) &= 1 - \frac{M}{r^6}, \\
    k(r) &= 1 + \frac{\beta^2 M}{r^4}
\end{align*}
\]

\( \beta \to 0 \): Schwarzschild black string

\( M \to 0 \): vacuum plane wave
We call this solution generating technique the **Null Melvin Twist**

**Important disclaimer:**

- Null Melvin Twist generates spacetimes supported by the 3-form background field strength. Not the same as BMN background supported by the 5-form background field strength.

- Works for black strings but not for black holes (because we need at least one isometry direction to do the twist)

Nonetheless, very interesting solution generating technique
What might one say about the physical properties of the black hole?

- **Area of the horizon**
  
  This is easy, and comes out be independent of $\beta$.

- **Surface gravity**
  
  This is subtle, because this depends on the choice of normalization of the time like Killing vector.

  If we take the time-like Killing vector to be $\frac{\partial}{\partial t}$, then the answer is again $\beta$ independent.
String/Black hole correspondence principle:

Entropy of a black hole \( S_{BH} = \frac{1}{G_9} (G_9 M)^{7/6} \)

Entropy of a string \( S_s = l_s M \)

match regardless of \( g_s \)

\[ S_{BH} = S_s = \frac{R_9}{g_s^2 l_s} \]

when the horizon radius is comparable to the string scale \( R_H^6 \sim G_9 M \sim l_s^6 \)
In plane waves, the string entropy is changed

\[ S_s = \frac{l_s}{1 + 8\pi l_s \beta} \]

(Pando-Zayas, Vaman; Greene, Schalm, Shiu; Brower, Lowe, Tan)

Black string entropy is unchanged, so the matching point shifts to

\[ R_H = \frac{l_s}{1 + 8\pi \beta l_s} \]

Correspondence principle broken?