Aspects of hadronic physics
in the gauge/gravity correspondence

Based on:

– *Hadronic Density of States from String Theory*,
  with D. Vaman.

– *Regge Trajectories Revisited in the Gauge/Gravity Correspondence*, hep-th/0310nnn
  with J. Sonnenschein and D. Vaman
Motivation (Wall breaker):

- AdS/CFT: Beyond the Supergravity Approximation. Sugra modes ↔ Protected Operators.

- Sectors of Large Charge:
  
  BMN: Large R-charge in $\mathcal{N} = 4$ SYM ↔ Strings in RR plane wave Background.
  
  GKP: Twist-two operators ↔ folded string spinning in $AdS_5$.
  
  PS: Scattering can be treated semiclassically by convolution of wave functions.

Sectors of large charges can be described by semiclassical string configurations.

What is a good universal quantum number?

Can we find out anything about $\mathcal{N} = 1$ SYM? $U(1)_R$ is broken!

Can the density of States be computed without full knowledge of the spectrum (without fully solving string theory)? [(confining) Kutasov]

High spin states, Regge trajectories. Hadronic states in Gauge/Gravity.
Outline

• Conceptual Framework

• Story in Supergravity

• Hagedorn Density of States in String Theory

• Annulons and their Thermal Partition Function

• Semiclassical calculation of Thermal Partition Function
  (For Sugra Backgrounds dual to Confining Gauge Theories)

• (II) Regge trajectories revisited in the Gauge/Gravity Correspondence
Conceptual Take Home

Heroes of the Day:

Genus expansion: The intuition maker

\[ Z_{\text{string}} = e^{-2\Phi_0} Z_0 + Z_1 + e^{2\Phi_0} Z_2 + \ldots \]

\[ Z_{\text{string}} = N^2 Z_0 + N^0 Z_1 + \frac{1}{N^2} Z_2 + \ldots \]

- Conformal Theories: Main Contribution is \( N^2 \).
- Confining Theories: Main Contribution is \( N^0 \).
A proposal for a semiclassical evaluation of $Z_1$:

The Solitonic Object for Torus Topology World Sheet

\[ X^0 = n \beta \sigma_1 + m \beta \sigma_2, \quad \text{"Completion"} \]

Include quantum fluctuations around this soliton to compute $Z_1$:

\[ Z \approx Z_{\text{soliton}} Z_{\text{quantum}} \]

The talk (I): Motivation and Justification for this proposal.
Supergravity Story Predates AdS/CFT [Klebanov]

AdS/CFT Correspondence

\[ AdS_5 \times S^5 \iff \mathcal{N} = 4 \ SU(N) \ \text{SYM} \]

The Sugra limit \( N \gg (g_{YM}^2 N)^{1/4} \gg 1 \)

\[
ds^2 = h^{-1/2}(r)[ - f(r)dt^2 + dx^i dx_i ] + h^{1/2}[f(r)^{-1}dr^2 + r^2 d\theta^2_5] \]

\[ h(r) = \frac{R^4}{r^4}, \quad f(r) = 1 - \frac{r_0^4}{r^4} \]

Temperature: \( T = 1/\beta = \frac{r_0^2}{\pi R^2} \)

\[ S_{BH} = \frac{A_h}{4G} = \frac{\pi^2}{2} N^2 V_3 T^3 + \ldots \]

Free \( U(N) \ \mathcal{N} = 4 \) Supermultiplet

Content: Gauge Field, \( 6N^2 \) massless scalars, \( 4N^2 \) Weyl Fermions

\[ S_0 = \frac{2\pi^2}{3} N^2 V_3 T^3. \]

The Famous \( 3/4 \)

\[ S = N^2 f(g_{YM}^2 N) VT^3 \]
Why is a Hagedorn Density of States So Generic?

- Large N theories might be related to strings (‘t Hooft, Polyakov, Maldacena).
- Should we expect generic properties of strings to be manifest in the field theory?

String states and Virasoro algebra: Given a primary state $|\h\rangle$

$$L_0|h\rangle = h|h\rangle$$
$$L_m|h\rangle = 0, \quad m > 0.$$

States

$$L_{-k_1}L_{-k_2} \ldots L_{-k_m}|h\rangle.$$

$L_0$ eigenvalue $h_j + \Sigma_i k_i$.
The $n^{th}$ level is spanned by vectors with $\Sigma_i k_i = n$

Q: How many such states?

A: $p(n)$ – Number of ways of writing $n$ as the sum of positive integers.

Hardy-Ramanujan:

$$p(n) \sim \exp\left(\pi \sqrt{2n/3}\right)$$

For strings: $\alpha' m^2 \sim n \implies S \sim E$ [Hagedorn] (!!!!) Nonperturbative effects, finite $N$
Motivating the Proposal: Compactified Boson on a Torus

- Configurations with nonzero winding number Torus $T^2 = \mathbb{C}/\Gamma : z \sim z + \omega_1 \sim z + \omega_2$

  \[ \Phi(z + k\omega_1 + k'\omega_2) = \Phi(z) + \beta(km + k'n), \quad k, k' \in \mathbb{Z} \]

  $(m, n)$ Specifies a Topological configuration.

  \[ \Phi = \Phi_{cla}^{m,n} + \phi, \quad \Phi_{cla}^{m,n} = \beta \left( \frac{z}{\omega_1} \frac{m\bar{\tau} - n}{\bar{\tau} - \tau} - \frac{\bar{z}}{\omega_1^*} \frac{m\tau - n}{\bar{\tau} - \tau} \right) \]

  $\phi$– periodic.

  \[ S[\Phi_{cla}^{m,n}] = \frac{1}{2\pi} \int dzd\bar{z} \partial \Phi_{cla}^{m,n} \bar{\partial} \Phi_{cla}^{m,n} = \beta^2 \frac{|m\tau - n|^2}{8\pi \tau_2} \]

  Modular Invariance $\longrightarrow$ Sum over all sectors $(m, n)$

  \[ Z = \sum_{m,n} Z_{m,n} = \frac{\beta}{\sqrt{8\pi} \tau_2^{1/2} |\eta(\tau)|^2} \sum_{m,n} \exp \left( -\frac{\beta^2}{8\pi \tau_2} |m\tau - n|^2 \right). \]

  \[ Z = Z_{\text{quantum}} Z_{\text{soliton}} \]

**Q:** Was the factorization an artifact of flat space?

**Q:** How to generalize for curved background?
Is there a solvable string theory of hadronic states?

Using a Penrose-Güven limit in Confining backgrounds.
Generic properties of AdS Dual of confining theories
- End of Space.
- Wilson Loop shows confining behavior.

\[ T_s = \frac{1}{2\pi \alpha'} g_{tt}(r_0) \]

- \( g_{tt}(r_0) \neq 0 \).
- \( g_{tt}(r_0) \) has a minimum (J. Sonnenschein et al.)
The Maldacena-Núñez background

- N D5 branes wrapped on $S^2$.
- IR: $\mathcal{N} = 1$ SYM contaminated with KK.

$$ds^2_{str} = e^{\phi_D} \left[ dx_\mu dx^\mu + \alpha' g_s N (d\rho^2 + e^{2g(\rho)}(d\theta_1^2 + \sin^2 \theta_1 d\phi_1^2) + \frac{1}{4} \sum_a (w^a - A^a)^2 \right], \quad e^{2\phi_D} = \frac{e^{2\phi_{D,0}} \sinh 2\rho}{2e^{g(\rho)}}$$

$$H^{RR} = g_s N \left[ -\frac{1}{4} (w^1 - A^1) \wedge (w^2 - A^2) \wedge (w^3 - A^3) + \frac{1}{4} \sum_a F^a \wedge (w^a - A^a) \right]$$

$$e^{\phi_{D,0}} = \sqrt{g_s N}$$

$$e^{2g} = \rho \coth 2\rho - \frac{\rho^2}{\sinh^2 2\rho} - \frac{1}{4}$$

$$A = \frac{1}{2} \left[ \sigma^1 a(\rho) d\theta_1 + \sigma^2 a(\rho) \sin \theta_1 d\phi_1 + \sigma^3 \cos \theta_1 d\phi_1 \right]$$

$$a(\rho) = \frac{2\rho}{\sinh 2\rho}$$

$w^a$ - SU(2) left-invariant one-forms

Scales associated with the $\mathcal{N} = 1$ SYM dual of the MN background.

$$M^2_{gb} \sim M^2_{KK} \sim \frac{1}{g_s N \alpha'}, \quad T_s \propto M^2_{gb} (g_s N)^{\frac{3}{2}}.$$
The Penrose-Güven limit: Set up

Make the following change of variables

\[ dt = dx^0, \quad x^i \to \frac{1}{L} x^i, \quad \rho = \frac{m_0}{L} r, \]

\[ \theta_2 = \frac{2m_0}{L} v, \quad \phi_+ = \frac{1}{2}(\psi + \phi_2), \]

where \( L^2 = \sqrt{g_s N} \) and \( m_0 = \frac{1}{\sqrt{g_s N \alpha'}} \) is the glueball mass

\[ \hat{\phi}_1 = \phi_1 + \frac{1}{3} \phi_+ \quad \hat{\phi}_2 = \phi_2 - \phi_+. \]

\[ x^+ = t, \quad x^- = \frac{L^2}{2}(t - \frac{1}{m_0} \phi_+), \]
The Penrose-Güven Limit

$L \rightarrow \infty; m_0 \text{ fixed}$

\[ ds^2 = -2dx^+dx^- - m_0^2 \left( \frac{1}{9}z_1^2 + \frac{1}{9}z_2^2 + v_1^2 + v_2^2 \right)(dx^+)^2 + dx^2 + d\bar{z}^2 + dv_1^2 + dv_2^2. \]

- 4 massless direction: (three $x$’s from WV and one $z$).
- 2 directions ($v$) with mass $m_0$.
- 2 directions with mass $\frac{1}{3}m_0$.

\[ H^{RR} = -2m_0 \, dx^+ \wedge [dv_1 \wedge dv_2 + \frac{1}{3}dz_1 \wedge dz_2]. \]

- Fermions: 4 with mass $m_0/3$ and 4 with mass $2m_0/3$

The Hamiltonian is (Poincare time/Energy):

\[ H = -p_+ = i\partial_+ = E - m_0\left(-\frac{1}{3}J_1 + J_2 + J_\psi\right) = E - m_0 \, J, \]

\[ P^+ = -\frac{1}{2}p_+ = \frac{i}{2}\partial_+ = \frac{m_0}{\Omega}(\frac{1}{3}J_1 + J_2 + J_\psi) = m_0\frac{J}{\sqrt{g_sN}}. \]
The Annulon Hamiltonian

The light-cone Hamiltonian of the theory has the following simple form:

\[
H = \frac{P_i^2}{2P_+} + \frac{P_+^2}{2P_+} + \frac{1}{2\alpha'P_+} \sum_{n=1}^{\infty} n(N_n^i + N_{i4}^n) \\
+ \frac{1}{2\alpha'p^+} \sum_{n=0}^{\infty} \left( w_n^a(N_n^1 + N_n^2) + w_n^b(N_n^3 + N_n^4) \right) \\
+ \frac{1}{2\alpha'p^+} \sum_{n=0}^{\infty} \left( \omega_n^\alpha S_n^\alpha + \omega_n^\beta S_n^\beta \right).
\]

(2)

where \(i = 1, 2, 3, 4\), \(a = 5, 6\), \(b = 7, 8\), \(\alpha = 1, 2, 3, 4\) and \(\beta = 5, 6, 7, 8\); \(N\) and \(S\) are bosonic and fermionic occupation numbers

\[
w_n^a = \sqrt{n^2 + (m_0p^+\alpha')^2}, \quad w_n^a = \sqrt{n^2 + \frac{1}{9}(m_0p^+\alpha')^2}, \\
\omega_n^\alpha = \sqrt{n^2 + \frac{1}{9}(m_0p^+\alpha')^2}, \quad \omega_n^\beta = \sqrt{n^2 + \frac{4}{9}(m_0p^+\alpha')^2}.
\]

(3)

A string theory of hadrons

The Hamiltonian purely in Field Theory language

\[
H = \left[ \frac{P_i^2}{2m_0J} + \frac{T_s}{2m_0J} (\mathcal{N}_R + \mathcal{N}_L) \right] + \left[ \frac{T_s}{2m_0J} (H_0 + H_R + H_L) \right].
\]
Towards the MN Annulon partition function

- building blocks: Boson off criticality [Itzykson and Saleur].

\[ z_{lc}^{(0,0)}(\tau, m) = \int \mathcal{D}X \exp \left[ - \int d^2z \bar{X} \left( -\partial_z \partial_{\bar{z}} + m^2 \right) X \right], \tag{4} \]

Doubly periodic quantum boson \( z = \xi_1 + \tau \xi_2, \)

\[ X(\xi_1, \xi_2) = \sum_{n_1, n_2 \in \mathbb{Z}} X_{n_1, n_2} \exp[2\pi i(n_1 \xi_1 + n_2 \xi_2)] \]

\[ d^2z = d\xi_1 d\xi_2 \tau_2, \]

\[ \partial_z \partial_{\bar{z}} = \frac{1}{4\tau_2^2} \left( |\tau|^2 \partial_1^2 - 2\tau_1 \partial_1 \partial_2 + \partial_2^2 \right), \]

Explicit Gaussian integrals over \( X_{n_1, n_2} \)

\[ z_{lc}^{(0,0)}(\tau, \mu) = \left[ \prod_{n_1, n_2 \in \mathbb{Z}} \tau_2 \left( \frac{2\pi}{4\tau_2} \right)^2 \left| n_1 \tau - n_2 \right|^2 + m^2 \right]^{-1}. \]

Double Product \( \rightarrow \) Modular properties

\[ z_{lc}(-1/\tau, m|\tau|) = z_{lc}(\tau, m) \]

Fermionic Partition function [antiperiodic in \( \xi_1 \)]

\[ z_{lc}^{(1/2,0)}(\tau, \mu) = \prod_{n_1, n_2 \in \mathbb{Z}} \tau_2 \left( \frac{2\pi}{4\tau_2} \right)^2 \left| \frac{2n_1 + 1}{2} - \frac{n_2}{2} \right|^2 + \frac{\mu^2 \beta^2}{\tau_2^2} \]

Too Formal!
A Nonholomorphic Generalization of Dedekind $\eta(\tau)$

Performing one of the infinite products

$$z^{(0,0)}_{lc}(\tau, m) = \exp \left[ 2\pi \tau_2 \left( m/2 + \sum_{n=1}^{\infty} \sqrt{n^2 + m^2} \right) \right]$$

$$\left[ \prod_{n \in \mathbb{Z}} \left( 1 - \exp[2\pi i(\tau_1 n + i\tau_2 \sqrt{n^2 + m^2})] \right) \right]^{-1}.$$

Compare with

$$|\eta(\tau)|^2 = \exp \left[ -\pi \tau_2 / 6 \right] \left[ \prod_{n \in \mathbb{Z}} \left( 1 - \exp[2\pi i n(\tau_1 + i\tau_2)] \right) \right]^{-1} \quad (5)$$

$\zeta$-function regularization of the Casimir Energy

$$\gamma_0(m) = \frac{m}{2} + \sum_{n=1}^{\infty} \sqrt{n^2 + m^2} = \frac{m}{2} + \left[ -\frac{1}{12} + \frac{1}{2} m - \frac{1}{2} m^2 \ln(4\pi e^{-\gamma}) \right]$$

$$+ \sum_{n=2}^{\infty} (-1)^n \frac{\Gamma(n - \frac{1}{2})}{n! \Gamma(-\frac{1}{2})} \zeta(2n - 1) m^{2n},$$

$\gamma$ – Euler constant. Flat space limit ($m \to 0$)

$$\gamma_0(m) \longrightarrow \sum_{n=1}^{\infty} n = \zeta(-1) = -1/12.$$
The MN Annulon Partition Function

\[
Z(\beta, \mu) = \frac{\beta}{4\pi l_s} \int_0^\infty \frac{d\tau_2}{\tau_2^2} \int_{-1/2}^{1/2} d\tau_1 \sum_{r=1}^{\infty} \left[ 1 - (-1)^r \right] \exp\left(-\frac{\beta^2 r^2}{2\pi \alpha' \tau_2}\right)
\]

\[
\times \left[ \tau_2^{-1/2} |\eta(\tau)|^{-2} \right]^4 \quad 4 \text{ massless bosons}
\]

\[
\times \left[ z_{lc}^{(0,0)}(\tau, \frac{m_0 \beta r}{\tau_2}) \right]^{2} \quad 2 \text{ } m_0 \text{ bosons}
\]

\[
\times \left[ z_{lc}^{(0,0)}(\tau, \frac{m_0/3 \beta r}{\tau_2}) \right]^{2} \quad 2 \text{ } m_0/3 \text{ bosons}
\]

\[
\times \left[ z_{lc}^{(1/2,0)}(\tau, \frac{m_0/3 \beta r}{\tau_2}) \right]^{4} \quad 4 \text{ } m_0/3 \text{ fermions}
\]

\[
\times \left[ z_{lc}^{(1/2,0)}(\tau, \frac{2m_0/3 \beta r}{\tau_2}) \right]^{4} \quad 4 \text{ } 2m_0/3 \text{ fermions}
\]

\[+ \text{ stuff associated with a nonsupersymmetric ground state} \quad (6)\]
Hagedorn Temperature:

\[-\frac{T_s \beta_H^2}{2\pi} + \frac{2}{3} \pi - 4\pi \gamma_0(m_0\beta_H) - 4\pi \gamma_0\left(\frac{m_0}{3}\beta_H\right) + 8\pi \gamma_{1/2}\left(\frac{m_0}{3}\beta_H\right) + 8\pi \gamma_{1/2}\left(\frac{2m_0}{3}\beta_H\right) = 0.\]  \hspace{1cm} (7)

Limits: \(m_0 \to 0\) [IIB Strings in Flat Space]

Large \(m_0\) a lower dimensional theory with \(\beta_H = \frac{2\pi}{(\sqrt{3}T_s^{1/2})}\)

Density of States for the MN Annulons

\[S = \frac{2\pi}{\sqrt{3}} \frac{E}{\tilde{T}_s^{1/2}}.\]

\(\tilde{T}_s = T_s/J\). In general \(S(m_0, T_s/J)\).
Where are the Temporal Windings?

- Using Light-Cone, temporal coordinate gauged away, RR field.
- How can this result be turned into evidence for the proposal.

The above Partition function was written over the strip:

\[ E : \quad \tau_2 > 0, \quad -\frac{1}{2} < \tau_1 < \frac{1}{2}. \]

Generalizing a flat space result: Tiling.

\[
Z(\beta, \mu) = \frac{\beta}{4\pi l_s} \int \frac{d\tau_2}{\tau_2^2} \int d\tau_1 \sum'_{m,n} \prod_{n_1,n_2 \in \mathbb{Z}} \exp \left( -\frac{\beta^2 |m\tau + n|^2}{2\pi \alpha' \tau_2} \right) \\
\times \left[ \tau_2^{-1/2} |\eta(\tau)|^{-2} \right]^4 \\
+ \text{stuff associated with a nonsupersymmetric ground state} \quad (8)
\]

\( m, n \in \mathbb{Z}^* \)
Integration over the fundamental domain:

\[ \mathcal{F} : \quad |\tau| > 1, \quad -\frac{1}{2} < \tau_1 < \frac{1}{2}. \]

**Mixing** \( Z_{quantumZ_{soliton}} \): \((m, n)\) dependent masses:

\[ \mu \rightarrow \mu|m\tau + n| \]
Temporal winding modes as solitons

The World Sheet has Torus Topology:

\[ ds^2 = |d\sigma_1 + \tau d\sigma_2|^2 = d\sigma_1^2 + |\tau|^2 d\sigma_2^2 + 2(\text{Re } \tau)d\sigma_1 d\sigma_2. \]

String Action [bosonic]:

\[ I = \frac{1}{2\pi\alpha'} \int d\sigma_1 d\sigma_2 \sqrt{\gamma} g_{\mu\nu} \gamma^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu. \]

EOM [assumption: \( g_{\mu\nu}(r) \)]:

\[ \partial_\alpha (\sqrt{\gamma} \gamma^{\alpha\beta} g_{00} \partial_\beta X^0) = 0. \]

\[ \partial_\alpha \left( \sqrt{\gamma} \gamma^{\alpha\beta} g_{rr} \partial_\beta r \right) - \frac{1}{2} \partial_r g_{00} \left[ \sqrt{\gamma} \gamma^{\alpha\beta} \partial_\alpha X^0 \partial_\beta X^0 \right] = 0. \]
Looking for the winding Soliton:

\[ X^0 = m\beta\sigma_1 + n\beta\sigma_2, \quad r = r(\sigma_1, \sigma_2). \]

Complicated in General

\[ \exists r_0 : \quad g_{00}(r_0) \neq 0, \quad \partial_r g_{00}(r_0) = 0, \quad \partial_r g_{rr}(r_0) = 0 \]

\[ S_\beta(m, n) = \frac{1}{2\pi \alpha'} \int d\sigma_1 d\sigma_2 \left[ \beta^2 (n^2 + m^2 |\tau|^2 - 2(\text{Re} \tau) mn) g_{00} + g_{rr} (|\tau|^2 \dot{r}^2 + r'^2 - 2(\text{Re}) \dot{rr}') \right]. \]

\[ S_\beta(m, n) = T_s \frac{1}{\tau_2} \left[ \beta^2 (n^2 + m^2 |\tau|^2 - 2(\text{Re} \tau) mn) \right] = T_s \frac{\beta^2 |m\tau - n|^2}{\tau_2}. \]

- For confining backgrounds \( r = r_0 \) is a solution [String at the AdS Wall] \( T_s = g_{00}(r_0)/2\pi \alpha' \)
Sketch of Fluctuations I:

- $r_0$ is now $\tau = 0$.
- $e_3^2 + e_4^2 + e_5^2|_{\tau=0} = \frac{1}{2}d\Omega_3^2$ round $S^3$ with radius $1/\sqrt{2}$.
- $S^3(\theta, \phi, \psi)$ by fixing $\theta = \pi/2$ becomes $\mathbb{R}^3(y^1, y^2, y^3)$.
- $e^2g|_{\tau=0} \approx \tau^2 \rightarrow \tau$-direction with $S^2(\theta_1, \phi_1)$ into $\mathbb{R}^3(\tau^1, \tau^2, \tau^3)$

$$S_{2b} = S[X_{\text{classical}}, r = r_0] + \frac{1}{2\pi \alpha'} \int d\sigma_1 d\sigma_2 \sqrt{\gamma^\alpha_\beta} (\partial_\alpha X^a \partial_\beta X^a g_{00} + \alpha' g_s N g_{00}[\partial_\alpha \tau^i \partial_\beta \tau_i + \frac{1}{4} \partial_\alpha y^i \partial_\beta y_i]$$

$$+ \frac{4\beta^2}{9Im\tau^2} g_{00} |m\tau - n|^2 \tau^i \tau_i$$

(9)

where $a = 1, \ldots, 4$ and $i = 1, 2, 3$.
- The mass $(2/3)\beta \sqrt{\frac{1}{\alpha g_s N}} |m\tau - n|/Im\tau$.

$$S_{2f} = \frac{i}{2\pi \alpha'} \int \bar{\theta}^I (\sqrt{\gamma}^\alpha_\beta \delta^{IJ} - \epsilon^{\alpha\beta} \sigma_3^{IJ}) \partial_\alpha X^0 \Gamma^0 g_0 \epsilon_0 (\delta^{JK} \partial_\beta + \frac{1}{8 \cdot 3!} e^\phi \sigma_1^{JK} \Gamma^{\mu_1 \mu_2 \mu_3} F_{\mu_1 \mu_2 \mu_3} \partial_\beta X^0 \Gamma_0 \epsilon_0) \theta^K$$

(10)

$$F(3) = -\frac{1}{4} g_s N dy_1 \wedge dy_2 \wedge dy_3$$

Choose the $\kappa : \Gamma^+ \theta^I = 0$ or $\theta^1 = \theta^2$
Thermalized fermionic degrees of freedom in a given soliton sector, characterized by \((m, n)\) winding numbers

\[
\theta(\sigma_1 + 1, \sigma_2) = (-1)^m \theta(\sigma_1, \sigma_1) \\
\theta(\sigma_1, \sigma_2 + 1) = (-1)^n \theta(\sigma_1, \sigma_1).
\]  

(11)

\[
Z_{T2} = \sum_{m,n \in \mathbb{Z}} \beta^2 \frac{\beta}{2\pi l_s} \int_F d^2 \tau \frac{1}{Im \tau^2} e^{-\frac{\beta^2 g_0}{4\pi\alpha'} \frac{|m\tau - n|^2}{Im \tau}} z^b_{0,0}(\tau, 0)^5 z^b_{0,0}(\tau, M^2) = \frac{4}{9} \beta^2 \frac{|m\tau - n|^2}{Im \tau^2} \frac{1}{\alpha' g_s N} z^f_{b_1,b_2}(\tau, 0)^8
\]

(12)

\[
z^b_{0,0}(\tau, M) = e^{-\pi Im \tau} \sum_{l \in \mathbb{Z}} \sqrt{l^2 + M^2} \prod_{l \in \mathbb{Z}} \left(1 - e^{-2\pi Im \tau \sqrt{l^2 + M^2 + 2\pi i Re l}}\right)^{-1}
\]

(13)

\[
z^f_{b_1,b_2}(\tau, M) = e^{\pi Im \tau} \sum_{l \in \mathbb{Z}} \sqrt{(l+b_1)^2 + M^2} \prod_{l \in \mathbb{Z}} \left(1 - e^{-2\pi Im \tau \sqrt{(l+b_1)^2 + M^2 + 2\pi i Re (l+b_1) - 2\pi i b_2}}\right),
\]

(14)

denotes the contribution of a GS fermion, with mass \(M\) and in the soliton sector \(m, n\), with twisted boundary conditions \(b_1 = (1 - (-1)^m)/2\) and \(b_2 = (1 - (-1)^n)/2\).
Hagedorn Behavior

- Partition function:

\[
Z_{T^2} \approx \int e^{-\beta^2 g_{00} I_m \tau} e^{-\pi I_m \tau \sum_{l \in \mathbb{Z}} (5l+3)\sqrt{l^2 + \frac{4}{9} \beta^2 \frac{1}{\alpha' g_s N}} - 8(l+\frac{l}{2})},
\]  

\[T_H:\]

\[
\frac{1}{4\pi\alpha'} \beta_H^2 g_{00} = -2\pi \left(5\gamma_0(0) + 3\gamma_0 \left(2\beta_H \frac{1}{\alpha' g_s N/3} \right) - 8\gamma_{1/2}(0)\right).
\]

\[
d(E) \approx \exp \left(\sqrt{3\pi} \frac{E}{T_s^{1/2}} \right).
\]

The Density of states depends on the gauge theory quark-antiquark string tension
Outlook:

- Exact Calculation of the Density of States in Hadronic String Theories.

- A proposal for how to compute the Hagedorn Density of States when the full string solution is not available.

- How about transitions: Confinement/Deconfinement?

- What other hadronic properties can one get a handle on?
Regge Trajectories Revisited in the Gauge/Gravity Correspondence

• A Regge trajectory: a line in the Chew-Frautschi plot: \( J = \alpha_0 + \alpha' t \)

• Well described by simple strings model but now we have the right string models.

<table>
<thead>
<tr>
<th>Gauge Theory State</th>
<th>String Theory Configuration</th>
</tr>
</thead>
<tbody>
<tr>
<td>Glueballs</td>
<td>Spinning Folded Closed String</td>
</tr>
<tr>
<td>Mesons of heavy quarks</td>
<td>Spinning open strings</td>
</tr>
<tr>
<td>Baryons of heavy quarks</td>
<td>Strings attached to a baryonic vertex</td>
</tr>
<tr>
<td>Dibaryons</td>
<td>Strings attached to wrapped branes</td>
</tr>
</tbody>
</table>

Table 1: States in gauge theory and their corresponding classical configuration in the string theory.
Closed spinning strings in supergravity backgrounds
\[ ds^2 = h(r)^{-1/2} \left[ -dt^2 + dx_1^2 + dx_2^2 + dx_3^2 \right] + h(r)^{1/2}dr^2 + \ldots \] (18)

The relevant classical equations of motion are

\[ \partial_a(h^{-1/2} \eta^{ab} \partial_b t) = 0, \]
\[ \partial_a(h^{-1/2} \eta^{ab} \partial_b x^i) = 0, \]
\[ \partial_a(h^{1/2} \eta^{ab} \partial_b r) = \frac{1}{2} \partial_r(h^{-1/2}) \eta^{ab}[-\partial_a t \partial_b t + \partial_a x_i \partial_b x^i]. \] (19)

They are supplemented by the standard Virasoro constraints.

ansatz

\[ t = e \tau, \]
\[ x_1 = f_1(\tau) g_1(\sigma), \quad x_2 = f_2(\tau) g_2(\sigma), \]
\[ x_3 = \text{constant}, \quad r = r(\sigma). \] (20)

The above system can be greatly simplified by further taking

\[ f_1 = \cos e\omega \tau, \quad f_2 = \sin e\omega \tau, \quad \text{and} \quad g_1 = g_2. \] (21)

The energy and angular momentum

\[ E = \frac{e}{2\pi\alpha'} \int h^{-1/2} d\sigma, \] (22)
\[ J = \frac{e\omega}{2\pi\alpha'} \int h^{-1/2} g^2 d\sigma. \] (23)

• A spinning string in the Poincare coordinates is dual to a state of energy \( E \) and spin \( J \).
Closed spinning strings in confining theories

- Conditions for confinement in gauge/gravity: $g_{00}$ has a nonzero minimum at some point $r_0$.

\[ \partial_r (g_{00})|_{r=r_0} = 0, \quad g_{00}|_{r=r_0} \neq 0. \]  

\[ t = e^\tau, \quad x_1 = \frac{1}{\omega} \cos e\omega \tau \sin e\omega \sigma \quad x_2 = \frac{1}{\omega} \sin e\omega \tau \sin e\omega \sigma \]  

\[ E = \frac{g_{00}(r_0)}{2\pi \alpha'} \int_0^{\bar{g}_0} \frac{d\bar{g}}{\sqrt{1 - \bar{g}^2}} = \frac{g_{00}(r_0)}{2\pi \alpha'} \arcsin \bar{g}_0 \]  

\[ J = \frac{g_{00}(r_0)}{2\pi \alpha'} \frac{1}{\omega^2} \int_0^{\bar{g}_0} \bar{g}^2 \frac{d\bar{g}}{\sqrt{1 - \bar{g}^2}} = \frac{g_{00}(r_0)}{2\pi \alpha'} \frac{1}{\omega^2} \frac{\arcsin \bar{g}_0 - \bar{g}_0 \sqrt{1 - \bar{g}_0^2}}{2}, \]  

$\bar{g}_0 \to 1$ [Ends spinning at the speed of light]

\[ E = \frac{g_{00}(r_0)}{2\pi \alpha'} \int_0^{1} \frac{d\bar{g}}{\sqrt{1 - \bar{g}^2}} = \frac{g_{00}(r_0)}{2\pi \alpha'} \frac{\pi}{2} \]  

\[ J = \frac{g_{00}(r_0)}{2\pi \alpha'} \frac{1}{\omega^2} \int_0^{1} \bar{g}^2 \frac{d\bar{g}}{\sqrt{1 - \bar{g}^2}} = \frac{g_{00}(r_0)}{2\pi \alpha'} \frac{1}{\omega^2} \frac{\pi}{4}. \]  

Typical Regge trajectories

\[ E^2 = 4\pi T_s \ S. \]  

\[ (24) \]

\[ (25) \]

\[ (26) \]

\[ (27) \]

\[ (28) \]
Semiclassical quantization

• Quadratic fluctuations in flat space.

• New feature of confining backgrounds?

\[ \gamma^{\tau\tau} g_{tt} \partial_{\tau} t \partial_{\tau} t + \gamma^{\alpha\beta} \partial_{\alpha} x^i \partial_{\beta} x^i g_{ii} = \left( \frac{8e^{\phi_0}}{9} \kappa^2 \cos^2 \omega \sigma \right) \tau^i \tau_i \]  \hspace{1cm} (29)

\[ [\partial^2_{\tau} - \partial^2_{\sigma} + m_0^2 \cos^2(\omega \sigma)] \delta \tau_i = 0. \]  \hspace{1cm} (30)

where \( i = 1, 2, 3 \) and \( m_0^2 = \frac{8e^{\phi_0}}{9} \kappa^2 \) for the MN solution and \( m_0^2 = \frac{e^{4/3}}{g_s M \alpha'} a_1^2 \frac{a_0}{3} \frac{\kappa^2}{\alpha'} \) for the KS solution.

This equation can be put in the standard way of Mathieu differential equation

\[ \frac{1}{\omega^2} \frac{d}{d\sigma^2} \left[ \frac{d^2}{d\sigma^2} + \frac{1}{\omega^2} (n^2 - \frac{m_0^2}{2}) - \frac{m_0^2}{2\omega^2} \cos 2\sigma \right] \delta \tau_i(\sigma) = 0. \]  \hspace{1cm} (31)

\[ \lambda_{r,n} = \frac{n^2}{\omega^2} + \frac{m_0^2}{2\omega^2} + \frac{r^2}{\omega^2} + \frac{1}{2(r^2 - 1)} \frac{m_0^4}{16\omega^4} + \mathcal{O}(m_0^8), \]  \hspace{1cm} (32)

where \( r \) and \( n \) are integers

\[ \Delta E = -\frac{1}{12} + \frac{1}{\sqrt{2} m_0}. \]  \hspace{1cm} (33)