The Kontsevich Model and the geometry of the moduli space of FZZT branes

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+ unpublished work
A non-trivial toy model for string theory: non-critical closed bosonic strings with $c<1$

1. World sheet description:
   $(p,q)$ minimal conformal matter coupled to Liouville CFT ($C_{(p,q)} + C_L = 26$)
   hard to compute

2. Random triangulations of world sheet:
   - Double-scaled matrix models
     recursion relations for closed string amplitudes

3. Solution of recursion relations via matrix model Feynman diagrams
   (no double-scaling):
   - Generalized Kontsevich matrix models
The modern perspective: D-branes

The world sheet theory admits 2 types of D-branes

(a) ZZ branes: "Dirichlet" in the Liouville direction - non-stable

(b) FZZT branes: "Neumann" in the Liouville direction - stable

2) is the effective world volume theory on N ZZ branes

3) is the open string field theory on N FZZT branes

the equivalence between 1) and 3) is open/closed duality
In this talk:

I will focus on the OSFT interpretation of Koutsevich model for (3,1) non-critical bosonic strings

(i) Koutsevich model makes predictions for open string amplitudes

$k$-point amplitude is given by a covariant derivative of $k-1$-point amplitude

the connection captures contact terms between open vertex operators

(ii) Comparison of open string amplitudes computed via Koutsevich model and via world sheet description (Seiberg and Shih)

supports the identification of Koutsevich model with the OSFT of FZZT branes

such identification has been derived from the world sheet formulation only for (1,2) model (Garotto - Rastelli)

a derivation of Koutsevich from double-scaled matrix models for the general (3,1) case has been done by Hashimoto, Huang, Kleban, Shi
iii) background independence in Kontsevich OSFT

the Kontsevich "partition function" is a half-density on the moduli space of FZZT branes

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Kontsevich model for (P11) non-critical strings (at \( \mu = 0 \))

\[ e^F(g_s, Z) = \int dM \ e^{-\frac{1}{g_s} S_K(M, Z)} \]

\[ S_K(M, Z) = \text{tr} \left[ V(M) - V(Z) - V'(Z)(M-Z) \right] \]

\[ V(M) = \frac{M^{p+1}}{p+1} \]

M: \( N \times N \) hermitian matrix field

Z: \( N \times N \) hermitian external matrix source

with eigenvalues \( \{ z_1, \ldots, z_N \} \)
Egs. \( p = 2 \quad V(M) = \frac{M^3}{3} \)

\( (2,1) \) model \( \leftrightarrow \) topological gravity

\[ e^F(q_s, Z) = e^{-\frac{2}{3} \frac{1}{q_s} \text{tr}Z^3} \int dM \ e^{-\frac{1}{q_s} \text{tr} \left( \frac{M^3}{3} - Z^2 M \right)} \]

\[ = \int dX \ e^{-\frac{1}{q_s} \text{tr} \left[ ZX^2 + \frac{X^3}{3} \right]} \]

\( M = Z + X \)

in terms of \( X \) the Kontsevich action looks like cubic OSET

**Propagator**

\[ \Delta^{(Z)}_{k, ij} = q_s \frac{\delta_{ij} \delta_{k}}{Z_i + Z_j} \]

\[ i = k \]

\[ j = l \]

\[ : \frac{q_s}{Z_i + Z_j} \]

**Cubic Vertex**

\[ \text{\( \frac{1}{q_s} \)} \]
$F(g_s, z)$ generates connected vacuum correlators of the OSFT which corresponds to triangulations of closed Riemann surfaces of genus $g$ and $k$ faces.

Open/closed duality relates $F(g_s, z)$ to the generating functional of closed string amplitudes of non-critical string theory. More precisely, if $\frac{1}{2} On$ are closed observables

$$\langle e^{2\pi i n \cdot On} \rangle \mid_{tn} = \frac{gs}{\hbar} Z^{-n}$$

$$= N^{-1} (Z) e^{F(g_s, z)} \quad (\ast)$$

$$N (Z) = \int dx e^{-\frac{gs}{\hbar} \int x^2}$$

The equality $(\ast)$ is valid for any finite $N$ $tn$ with $n = 1, \ldots, N$ are independent parameter.

For $N$ large enough one recovers the full generating functional of closed amplitudes.
Eqs: 2-loops

\[ \frac{1}{6} \sum_{i,j,k} \frac{\frac{g_s}{2i2j2k}}{g_s^2} = \frac{1}{6} \frac{t_1^3}{g_s^2} = \frac{1}{3} \langle Q_1^3 \rangle_{g=0} t_1^3 \]

\[ \Rightarrow \langle Q_1^3 \rangle_{g=0} = 1 \]

In the modern perspective (Kontsevich = OSFT),
correlators of Kontsevich matrix theory with
n-external legs

have a physical meaning: they compute open string amplitudes of
n vertex operators on an open Riemann surface with FZZT boundary conditions.

i) Which vertex operators?

ii) What is the world sheet meaning of \( Z \)?
Brief review of FZZT Liouville branes

\[ S_L = \int \mathcal{A} \Phi \overline{\Phi} + \mu e^{2b\Phi} + \int \frac{\mu_B e^{b\Phi}}{2\Sigma} \]

\[ c_L = 1 + 6Q^2 \quad , \quad Q = b + \frac{1}{b} \quad , \quad \left| b^2 = \frac{\Phi}{\Phi} \right| \]

FZZT boundary conditions:

\[ \partial_n \Phi \bigg|_{\partial_2} = \mu_B e^{b\Phi} \]

Closed vertex operators:

\[ \nu_x = e^{2x\Phi}, \quad \Delta_x = x(Q-x) \]

Open vertex operators:

\[ \beta = e^{\beta\Phi}, \quad \Delta_{\beta} = \beta(Q-\beta) \]

\[ V_b = e^{2b\Phi} \] is a marginal closed operator

\[ \Rightarrow \text{bulk cosmological constant} \mu \]

\[ B_b = e^{b\Phi} \] is a marginal open operator

\[ \Rightarrow \text{boundary cosmological constant} \mu_B \]

with N FZZT branes: the values of \( \mu_B \) on

The \( i \)-th brane \( \mu_B^{(i)} \) parametrize the open moduli space
Garotto-Rastelli conjecture:

In the $(1,2)$ model

(i) The vertex operator corresponding to the string field $X_{ij}$ is $(Bb)_{ij}$

(ii) $Z^{(i)} \equiv \mu_b^{(i)}$ \hspace{1cm} (*)

N.B. Liouville Theory has a symmetry under:

\[ b \leftrightarrow \frac{1}{b} \quad (\text{or} \quad p \leftrightarrow q), \quad \mu \leftrightarrow \hat{\mu}, \quad \mu_b \leftrightarrow \hat{\mu}_b \]

where \( \hat{\mu} = \mu^{1/b^2} \) is the dual bulk cosmological constant.

\[ \hat{\mu}_b = \langle Bb \rangle_{Disc} \] is the dual boundary cosmological constant.

An equivalent statement to (X) is in the $(2,1)$ model

(i) $X_{ij} \leftrightarrow B_{ij}$ (but not $Bb$)

(ii) $Z^{(i)} \leftrightarrow \hat{\mu}_b^{(i)}$ (but not $\hat{\mu}_b$)

(more on the consistency between $(p,q)$ and $(q,p)$ \hspace{1cm} LATER )
Using the G-R correspondence one can compare open string amplitudes computed in the Kontsevich model with those computed in the world sheet formulation (essentially Disk amplitudes with \( N=1 \)) and make predictions for amplitudes which are hard to compute on the world sheet.

\[
\langle X_{i_1} \ldots X_{i_{n-1}} \rangle_{\text{connected-amputated}} \equiv \int_{g_{n-2}} (Bb)_{i_1 \ldots i_{n-1}} \equiv \int_{g_{n-2}} (Bb)_{i_1 \ldots i_{n-1}}
\]

Construct the generating functional of connected amputated amplitudes in the Kontsevich model

\[
e^{F_x(q_g, z, J)} = \int dx \ e^{-\frac{1}{q_g} \ \text{tr}[2x^2 + \frac{x^3}{3}] + \text{tr} J X}
\]

\( F_x(q_g, z, J) \) generates connected graphs\( \Rightarrow \) replace \( J \) with the solution of the linearized eqs. of motion

\[
J = \dot{z} + \phi
\]
\( F_x(q_s, z, \bar{z}, \phi_y) \) \text{ generates connected amputated graphs.}

A more "convenient" object (see later) is the effective potential:

\[
H(z, \phi) = F_x(q_s, z, \bar{z}, \phi_y) - \frac{1}{g_s} \text{tr} Z \phi^2 - \log N(z)
\]

Note: \( H(z, \phi = 0) = F(q_s, z) \) is the generator of vacuum amplitudes considered earlier.

For a generic QFT \( H(z, \phi) \) cannot be derived from \( H(z, 0) \).

In our case, however:

\( \phi \)-derivatives of \( H(z, \phi) \) correspond to insertions of \( X \).

According to the correspondence \( X \leftrightarrow B_b \), insertions of \( X \) are equivalent to insertions of the marginal operator \( \oint A B_b \).

Insertions of \( \oint A B_b \) are given by \( \mu_B \)-derivatives up to contact terms:

\[
\mu_B \langle \ldots \rangle = \langle \oint B_b \ldots \rangle + \text{contacts}
\]
According to the correspondence \( z \leftrightarrow \mu_b \), \( \mu_b \)-derivatives are equivalent to \( z \)-derivatives.

**Conclusion:**

Where it not for contact terms one would expect

\[ \Theta \Phi H \sim \Theta z H \]

We will derive, from the second quantized Kontsevich model, the precise differential equation relating \( \Theta \Phi H \) and \( \Theta z H \) and show that it uniquely fixes the values of the contact terms (difficult to compute from world sheet).

**The Computation:**

Go back to \( m \) variable \( (M = x + z) \)

\[
\begin{align*}
F(x, z, J) &= e^{-\frac{1}{2} \frac{1}{g_s} \text{Tr} z^2 - \frac{1}{4} \text{Tr} jz} \int dM e^{-\frac{1}{g_s} \text{Tr} M^3 - (z^2 + j)^2} \\
&= e^{F(g_s, (z^2 + j)^{1/2})} + \frac{2}{3 g_s} \text{Tr} [(z^2 + j)^3 - z^3] - \text{Tr} jz
\end{align*}
\]

Vacuum generating functional
\[ H(z, \phi) = H((z^2 + \imath z_1 \phi_3)\frac{1}{2}, 0) \]

\[ + \log \frac{N((z^2 + \imath z_1 \phi_3)^{1/2})}{N(z)} \]

\[ + \frac{2}{3\phi_3} \text{tr}\left[ (z^2 + \imath z_1 \phi_3)^{3/2} - z_1^3 - 3z^2 \phi - \frac{3}{2} z \phi^2 \right] \]

Essentially:

\[ H(z, \phi) \sim H(z_1, 0) \]

\[ z_1' = (z^2 + \imath z_1 \phi_3)^{1/2} \]

Note: Naively would have been \( z_1' = z + \phi \)

This can be expressed in differential form as

\[ \left[ \frac{\partial}{\partial z_i} - \frac{\partial}{\partial \phi_{ij}} - \Gamma^{(mn)}_{(ij)(ke)} \phi_{mn} \frac{\partial}{\partial \phi_{ke}} + \frac{\partial}{\partial z_j} \log N(z) - \frac{1}{2\phi} \phi_{ij}^2 \right] e^H = 0 \quad (\star) \]

This equation contains information on a connection \( \Gamma^{(mn)}_{(ij)(ke)} \).
\[ \Pi^{(m)}_{(ij)(ke)} = \frac{1}{z_i + z_n} \delta_{jm} \delta_{ki} \delta_{en} + (a) \]
\[ + \frac{1}{z_m + z_j} \delta_{in} \delta_{ej} \delta_{km} \quad (b) \]

\( \Pi \) represents the two possible ways the vertex operators \( X_{ji} \) and \( X_{nm} \) can collide to give \( X_{ek} \).

(a) \[ \begin{array}{c}
\cdots \\
\text{\vdots} \\
\text{\vdots} \\
\text{\vdots} \\
\end{array} \]

(b) \[ \begin{array}{c}
\cdots \\
\text{\vdots} \\
\text{\vdots} \\
\text{\vdots} \\
\end{array} \]

By taking derivatives of \( (\star) \), it is easy to show that

\[ \langle X_{ji} \cdots X_{jn} \rangle_{c.a.} = \frac{D}{DZ_{ij}} \langle X_{ji} \cdots X_{jn} \rangle_{c.a.} \]

\[ \frac{D}{DZ_{ij}} = \frac{\partial}{\partial Z_{ij}} - \Gamma^{(ij)} \]

"bulk" "contacts"
Symmetry of correlators under exchange of two vertex operators requires the flatness condition:

\[
\left[ \frac{D}{Dz_{ij}}, \frac{D}{Dz_{ke}} \right] = 0
\]

That this is true follows from the fact that the covariant derivatives can be written as:

\[
\frac{D^{(m)}}{Dz_{ke}} \langle X_{ji}, \ldots X_{jmn} \rangle = \Delta^{-1}_{ji} a, b, \ldots \Delta^{-1}_{jmn} a, b, n \times
\]

\[
\times \frac{\partial}{\partial z_{ke}} \left[ \Delta_{abc; c, d} \ldots \Delta_{abu; cndn} \langle X_{cd}, \ldots X_{cdn} \rangle \right]
\]

Comparison with world sheet computations (Seiberg–Shih)

Take \( N=1 \) and \( \Sigma = \text{Disk} \) (tree level graphs of the OSFT)

In this case \( \Delta = \frac{1}{2z} \)

And

\[
\frac{D^{(m)}}{Dz} = \Delta^{-n} \frac{\partial}{\partial z} \Delta^n = \partial z - \frac{n}{z}
\]

\( \Delta \) when acting on \( n \)-point amplitude
The first diagram contributing to $H(z, \phi)$ at tree level is:

$$3 \quad \begin{array}{c} \quad 2 \\ 1 \end{array} + \quad \begin{array}{c} \quad 2 \\ 1 \end{array} = -\frac{2}{g_s}$$

**NOTE:** 0, 1, 2 point functions on the Disk are not defined. This was the reason to subtract the free part $\frac{1}{2} \text{tr} \ Z \phi^2$ in the definition of $H(z, \phi)$.

Analogously, the 0-point amplitude on the torus is not defined, and this is the reason for subtracting $\log N$.

\[\Rightarrow \quad \langle X^3 \rangle_{\text{tree}} \quad \text{c.a.} = -2 = \langle B_b(x_1) ... B_b(x_3) \rangle_{\text{Disk}}\]

If n-point amplitudes where given by simple $z$-derivatives they would vanish (for $n \leq 4$).

The only non-trivial contribution to higher point amplitudes comes from contact terms.

\[\langle \int_{Dz} B_b ... \int_{Dz} B_b \quad B_b(x_1) ... B_b(x_3) \rangle = \langle X^n \rangle_{\text{tree}} \quad \text{c.a.} \]

\[= D_z^{n-3} \langle X^3 \rangle_{\text{tree}} \quad \text{c.a.} \]

\[= D_z^n F_k(z) \quad \text{with} \quad F_k(z) = \frac{2}{3} z^3\]
One can rewrite the previous result in an equivalent way:

Introduce a new coordinate $x$ s.t.

$$\frac{dx}{dz} = \Delta^{-1} = 2z \quad \Rightarrow \quad x = z^2 + \text{const.}$$

Then

$$D_z^{(n)} = \Delta^{-n} \partial_z \Delta^n = \Delta^{-(n+1)} \partial_x \Delta^n$$

and

$$D_z^n = D_z^{(n-1)} D_z^{(n-2)} \ldots$$

$$= \Delta^{-n} \partial_x \Delta^{n-1} \Delta^{-(n-1)} \partial_x \Delta^{n-2} \ldots$$

$$= \Delta^{-n} \partial_x^n$$

In the coordinate $x$ the connection $\Gamma$ is zero

$$D_x \equiv \partial_x$$

The $n$-point amplitude can then be written as

$$\langle X^n \rangle_{\text{tree}}^{\text{c.a.}} = D_z^n F_k(z)$$

$$= \Delta^{-n} \partial_x^n F_k(z; x)$$

$$= \left(\frac{dx}{dz}\right)^n \partial_x^n F_k(z; x)$$

Under a reparametrization $z \rightarrow x$ of the open moduli space, correlators transform as tensors.
COMPARISON WITH WORLD SHEET RESULTS

Seiberg and Shih remarked that correlators of $B_6$ in $(p,q)$ minimal strings are captured by the curve

$$T_p(y) - T_q(x) = 0 \quad (*)$$

(where $T_p(\cos \theta) = \cos(p\theta)$)

according to the following recipe:

let $y_{p,q}(x)$ be the solution of $(*)$

and $F_{p,q}(x) = \int^x dx' y_{p,q}(x')$

Then:

$$\left\langle \left( \int_{qz} B_6 \right)^n \right\rangle = \frac{\partial^n}{\partial x^n} F_{p,q}(x)$$

with $x = \mu B$

($\mu = 1$ here)

Note that $\langle B_6 \rangle = \frac{\partial}{\partial x} F_{p,q}(x) = y \equiv \hat{\mu}_B$

The ground ring relation $(*)$ is invariant under $p \leftrightarrow q$ and $x \leftrightarrow y$ (or $\mu_B \leftrightarrow \hat{\mu}_B$)

in agreement with the symmetry of Liouville
However, had we exchanged \( p \leftrightarrow q \) the S-S recipe would have given a different result.

Eqs. \((p,q) = (2,1)\)

\[
T_2(y) = y^2 - 1, \quad T_1(x) = x
\]

The curve \((\ast)\) is:

\[
y^2 - 1 = x \quad \Rightarrow \quad y = \sqrt{1+x}
\]

and

\[
F_{2,1}(x) = \int_{x}^{1} y \, dy' = \frac{2}{5} (1+x)^{3/2} = \frac{2}{5} y^3
\]

according to \((S-S)\) the correlators are

\[
\langle (f_{22} B_6)^n \rangle_{2,1} = \Theta^n_x F_{2,1}(x) = \Theta^n_x \left[ \frac{2}{5} (1+x)^{3/2} \right]
\]

The \((S-S)\) recipe for the \((1,2)\) model would have been, however

\[
\langle (f_{22} B_6)^n \rangle_{1,2} = \Theta^n_y F_{1,2}(y)
\]

with

\[
F_{1,2}(y) = \int_{y}^{1} dy' x = \frac{y^3}{5} - y
\]

and thus

\[
\langle (f_{22} B_6)^n \rangle_{1,2} = \left\{ \begin{array}{ll}
2 & \text{for } n = 3 \\
0 & \text{for } n \geq 4
\end{array} \right.
\]

N.B. \( x = \overline{\mu}_B^{(2,1)} = \overline{\mu}_B^{(1,2)} \)

\( y = \overline{\mu}_B^{(2,1)} = \overline{\mu}_B^{(1,2)} \)
According to G-R we should compare with the (1,2) model and 
\[ Z = \mu_B^{(1,2)} = \mu_B^{(2,1)} = y \]

- There is agreement for \( n=3 \)
- There seems to be disagreement for \( n \geq 4 \) : in the (1,2) case the (S-S) recipe does not take into account contact terms (in \( z=y \) coordinate the connection is not zero!)

- The (S-S) recipe is complete for the (2,1) model (in \( x \) coordinate the connection is zero and contact terms vanish)

To compare Kontsevich correlators with (2,1) model correlators we should take into account that going from (1,2) to (2,1) involves a change of coordinates \( z=y \rightarrow x \) under which correlators transform as tensors...
Kontsevich:

\[ \langle X^n \rangle_{\text{tree \ c.a.}} = \left( \frac{dx}{dz} \right)^n \frac{\partial^n}{\partial x^n} \left[ \frac{2}{3} Z(x)^3 \right] \]

\[ Z(x) = (1+x)^{\frac{1}{2}} \]

World-Sheet (S-S):

\[ \langle \left( \int_{\partial \Sigma} \mathcal{B}_b \right)^n \rangle_{(2,1)} = \frac{\partial^n}{\partial x^n} \left[ \frac{2}{5} (1+x)^{3/2} \right] \]

\[ \langle X^n \rangle_{\text{tree \ c.a.}} = \langle \left( \int_{\partial \Sigma} \mathcal{B}_b \right)^n \rangle_{(1,2)} \]

\[ = \left( \frac{dx}{dz} \right)^n \langle \left( \int_{\partial \Sigma} \mathcal{B}_b \right)^n \rangle_{(2,1)} \]

Agreement
Generalization To (p|1) Models

curve : \( T_p(y) - x = 0 \)

where \( T_p(y) \rightarrow y^p \) for \( \mu \rightarrow 0 \)

One can (easily) see that the (S-S) recipe for \( n \)-point open string amplitudes of \((p|1)\) string agrees with connected amputated correlators of a generalized Kontsevich model with potential

\[
V'_p(M) = T_p(M) \quad \Rightarrow \quad V_p(M) = \frac{\mu^{p+1}}{p+1}
\]

(for \( \mu \rightarrow 0 \))

again with the identification

\[
z = \mu^{\frac{1}{(p)}} \quad \text{and} \quad X \leftrightarrow Bb \quad (b = \sqrt{I_p})
\]

The Kontsevich connection is \( \Pi = \frac{V'''}{V''} \)

Note however that in this case the Kontsevich action for \( X = M - Z \) is:

\[
S_k(X, Z) = \frac{1}{p+1} \text{tr} \left[ (X + Z)^{p+1} - Z^{p+1} - (p+1)Z^pX \right]
\]

\[
= \frac{1}{2} \text{tr} \left[ X \Delta^{-1}(z) X + c_3 X^3 + \ldots + c_{p+1} X^{p+1} \right]
\]

contains interactions \( X^h \) for \( h = 3, \ldots, p+1 \)

\( \Rightarrow \) No Witten's cubic OSFT, but some sort of "effective" SFT
GEOMETRICAL PROPERTIES OF THE KOUTSEVICH EFFECTIVE POTENTIAL

As we have seen, $\Psi(z, \phi) = e^{H(z, \phi)}$ satisfies the differential equation

$$[\partial_z - (\partial_\phi + \Gamma_{zz} \phi^2 - \frac{1}{2} \partial_\phi \Gamma_{zz}^2)] \Psi(z, \phi) = 0$$

with

$$\Gamma_{zz} = A^{-1} \Gamma_z A$$

This equation expresses the background independence of $\Psi(z, \phi)$

- $z$ is a "base point" on the open moduli space $\mathcal{M}$
- $\phi \in T_z \mathcal{M}$ defines a system of local coordinates around $z$
- $\Psi(z, \phi)$ is the expression in local coordinates of a geometrical object $\rho(z')$

$$\Psi(z, \phi) \sim \rho(z') \text{ with } z' = (z^2 + t_z \phi)^{\frac{1}{2}}$$

However, $\rho(z')$ is not a scalar function of $z'$

Look at the covariance properties of (i) under

$$\begin{align*}
\hat{z} &= z(z) \\
\hat{\phi} &= \frac{\partial z}{\partial z} \phi
\end{align*}$$

under which $\Gamma$ transforms as a connection
(ii) is covariant if \( \Psi \) transforms as

\[
\Psi(\tilde{z}, \tilde{\phi}) = \Psi(z, \phi) \left( \frac{\partial \phi}{\partial \tilde{\phi}} \right)^{1/2}
\]

This means that \( \Psi(z, \phi) \) is the representation in local coordinates of a half-density

\[
\Psi(z, \phi) \, d\phi^{1/2} = \rho(z') \, dz'^{1/2}
\]

This is in agreement with various observations in the literature (Aganagic, Dijkgraaf, Klemm, Horava, Vafa, Maldacena, Moore, Seiberg, Shih, Aganagic, Neitzke, Vafa, etc.)

**Curvature of Kontsevich Connection**

Extending \( z \) to \( \phi \), the connection \( \Gamma = A^{-1} \partial A \) has non-zero \((1,1)\) curvature

\[
\overline{\partial} \Gamma \neq 0 \quad \text{(e.g., for \((2,1)\) model } \Gamma = \frac{1}{2}
\]

\[
\overline{\partial} \Gamma = \overline{\partial} \frac{1}{2} = 8(2)
\]

The curvature is supported at points where the kinetic term of the Kontsevich action vanishes — a non-perturbative effect.
The existence of a (1,1) curvature is associated with another differential equation satisfied by $\Psi(z, \phi)$: **THE RENORMALIZATION GROUP EQUATION**

\[
\left[ \partial_{\bar{z}} - \frac{1}{2} \bar{\partial} \Delta \frac{\partial^2}{\partial \phi^2} \right] \Psi(z, \phi) = 0 \quad (ii)
\]

\[
\downarrow
\]

\[
\partial_{\bar{z}} H - \frac{1}{2} \bar{\partial} \Delta \frac{\partial H}{\partial \phi} \frac{\partial H}{\partial \phi} - \frac{1}{2} \bar{\partial} \Delta \frac{\partial^2}{\partial \phi^2} H = 0
\]

\[
\downarrow
\]

\[
\partial_{\bar{z}} H = \bullet - \bullet + \bullet
\]

(iii) expresses the fact that $H$ depends on $\bar{z}$ only through the propagator $\Delta = \frac{1}{2\pi}$

**Observations:**

(i) and (iii) are formally identical to the "holomorphic anomaly" equations satisfied by the partition function of closed topological strings.

(ii) and (ii) can be recursively solved (either for $z \to \infty$ or $z \to 0$) and determine uniquely $\Psi(z, \phi)$.
consistency of (i) and (iii), i.e.

$[\overline{\nabla}^2 - \frac{1}{2} \overline{\nabla} \overline{A}^2, \, \overline{\nabla}^2 - (\partial \phi + \Gamma \phi \partial \phi) - \frac{1}{2} \overline{c} \phi^2 - \frac{1}{2} \Gamma \Gamma] = 0$

relates the term $\overline{\nabla}^2$ in (iii) with the term $\frac{1}{2} \Gamma \Gamma$ in (i).

$\Rightarrow$ There is a connection between the fact that $\Psi(z, \phi)$ is a half-density on moduli space and the fact that the Kostsevich connection is non-flat.

The geometrical meaning of (i) and (iii) has to be understood better.
CONCLUSIONS:

The identification of the Kontsevich model with the OSFT of F227 branes implies non-trivial predictions for open string amplitudes.

- **Contact terms** are encoded by a matrix connection on the open moduli space.

- The existence of this connection solves some puzzles on the consistency between (1,1) and (p,q) minimal strings.

- The connection is non-flat and this is related with the fact that the Kontsevich generating functional \( T(z, \phi) \) is a half-density on moduli space.