PROBLEM SET 3. SOLUTIONS

Magnetic braking: (15 pts) In class, we briefly discussed magnetic braking: the spindown of an object due to transfer of angular momentum to the outside medium by magnetic torques. In this problem, we revisit this process, describe it in a simple way, and apply the result to the halo of our Galaxy (although historically the main application has been to star formation theory).

1. Consider an object of mass $M$ and moment of inertia $I$ which is threaded by magnetic flux $\Phi$. Assume the magnetic field extends into a medium of density $\rho_e$ and has strength $B_e$, and that initially the fieldlines are straight. Let the object initially rotate as a solid body with angular velocity $\Omega_0$. Give an approximate description of the state of the magnetic field and gas above and below the object at time $t > 0$. **Hint:** This is much like the first problem in last week’s homework set. **Solution:** The rotation of the object launches Alfvén waves into the ambient medium. At time $t > 0$, the medium is undisturbed for $|z| > v_{Ae}t$ ($v_{Ae} \equiv B_e/(4\pi \rho_e)^{1/2}$). At $|z| < v_{Ae}t$, the gas is rotating with speed $\Omega(t - |z|/v_{Ae})$.

2. Invoke conservation of angular momentum to derive an approximate expression for the rate of change of the angular velocity $\Omega$ of the body. Use your result to estimate the time it takes $\Omega$ to decrease to half its initial value. This is known as the magnetic braking time $t_B$. Give a qualitative discussion of explaining the dependence of $t_B$ on the various physical parameters. **Solution:** The angular momentum $L$ is the sum of the angular momentum in the cloud and that of the rotating gas column. For simplicity, imagine that everything is rotating at the same rate and that the column of rotating gas has radius $R$. Then

$$L = I\Omega_0 = (I + \pi \rho_e R^4 v_{Ae} t)\Omega.$$ 

The rotation rate has fallen to half its original value when the moment of inertia of the rotating column is equal to that of the rotating body. That is,

$$t_B \sim \frac{I}{\pi \rho_e R^4 v_{Ae}} \propto \rho_e^{-1/2} \Phi^{-1} R^{-2}.$$ 

The braking time is longer in a low density medium because it takes longer for the moment of inertia of the gas column to build up. In-
creasing the magnetic field increases the rate at which new material is set into rotation, so it reduces $t_B$. If we write $I = 4\pi K \rho_b R^5 / 3$, where $K$ is a constant, and assume $B = B_e$, we can write

$$t_B \sim \frac{4}{3} K \left( \frac{\rho_b}{\rho_e} \right)^{1/2} \frac{R}{v_{Ac}}.$$

That is, the body is braked in one Alfvén crossing time, multiplied by the square root of the density ratio.

3. Our Galaxy has a hot, gaseous halo. A still open question is the extent to which the halo co-rotates with the disk. Suppose the halo has density $n = 10^{-3}$ cm$^{-3}$ and radius $10^4$ pc. Estimate the timescale on which the halo is set into rotation with the disk as a function of vertical field strength $B_z$. Calculate the magnitude of $B_z$ such that the halo and disk are brought into corotation over the age of the disk, which is about $5 \times 10^9$ yr. Discuss how you could estimate the strength of the vertical field by measuring the rotation rate of the halo. **Solution:** According to Parts 1 and 2, the halo co-rotates with the disk if the Alfvén travel time across it is less than the age of the disk, i.e

$$v_{Ac} > 10\text{km/s}/5 \times 10^9 \text{yr} = 2\text{km/s}.$$

For the number density of the halo given here, this is equivalent to $B > 3 \times 10^{-8}$ G. This is only 1% of the disk field, and would be difficult to detect directly. If the disk is observed to corotate, this would be good evidence for a magnetized halo.

**Magnetic windup: (20 pts)** One of the ingredients of dynamo theory is strengthening the magnetic field by wrapping up the lines. This problem allows you to calculate this explicitly. Consider a differentially rotating disk with angular velocity $\Omega(r)$. Let the magnetic field lie in the plane of the disk, and write it in terms of a vector potential $\hat{z}A(r, \theta, t)$ as

$$B = \nabla \times \hat{z}A(r, \theta, t).$$

1. Show that the magnetic induction equation can be “uncurled” to yield an evolution equation for $A$

$$\frac{\partial A}{\partial t} = -\Omega \frac{\partial A}{\partial \theta}.$$
Solution: The magnetic induction equation can be written as

$$\nabla \times \left( \frac{\partial}{\partial t} - \mathbf{v} \times \mathbf{B} \right) = 0.$$ 

This equation holds if

$$\frac{\partial \mathbf{A}}{\partial t} = \nabla \times \mathbf{v} \times \mathbf{B} + \nabla \phi,$$

where $\phi$ is a scalar function. In this case, $\nabla \phi$ would have only a $\hat{z}$ component, but $\partial z \equiv 0$. So, we can set $\phi = 0$. The result follows.

2. Let the initial value of $A$ be $A_0(r, \theta)$. Show that

$$A(r, \theta, t) = A_0(r, \theta - \Omega(r) t)$$

is a solution of the induction equation. Solution: In this problem, $\mathbf{v} = \theta \hat{r} \Omega$ and $\mathbf{B} = \hat{r} r^{-1} \partial \mathbf{A} / \partial \theta - \theta \partial \mathbf{A} / \partial r$. Working out the vector product gives

$$\frac{\partial \mathbf{A}}{\partial t} = -\Omega \frac{\partial \mathbf{A}}{\partial \theta}.$$

Any function $f(r, \theta, t)$ which has the form $g(r) h(\theta - \Omega(r) t)$ is a solution of this equation. The function given is the one which fits the initial conditions.

3. Write down the components of $\mathbf{B}$ in terms of $A$ for the initial condition $A_0 = R B_0 \cos \theta$. Describe qualitatively how the field changes with time. Solution: There’s a typo here; the $R$ should be lower case. We then have

$$B_r = -B_0 \sin (\theta - \Omega t)$$

$$B_\theta = -B_0 \cos (\theta - \Omega t) + B_0 r \frac{d\Omega}{dr} \sin (\theta - \Omega t).$$

This describes a field with a radial component that remains constant in its average amplitude and an azimuthal field that increases linearly in amplitude with time. This increase is due to stretching by differential rotation.

4. Plot $\mathbf{B}$ as a function of time for the Galaxy-like rotation profile $\Omega = \Omega_0 (R/r)$ for $0.5 \leq r/R \leq 2$. Go out at least 10 Galactic years as measured at $r/R = 1$. I prefer a polar plot, but a plot of $B_r$ and $B_\theta$
as functions of $r$ at fixed $\theta$ will do if you cannot find a way to plot in polar coordinates. Does this look like a good model for producing the field reversals seen in our Galaxy? **Solution:** Since $B \cdot \nabla A \equiv 0$, the fieldlines are lines of constant $A$. You all showed that there are more and more field reversals over time. We can see this analytically by solving for the distance $\Delta r$ over which the phase of $A$ changes by $\pi$ at fixed $\theta$. We have

\[
\theta - \Omega(r + \Delta r)t = \theta - \Omega(r)t + \pi.
\]

Taylor expanding gives

\[
\frac{\Delta r}{r} = \frac{\pi}{trd\Omega/dr}.
\]

For $\Omega$ as given, $\Delta r/r$ can be written in terms of the local rotation period as $\Delta r/r = P/2t$. At present, $t \sim 25P$, so there should be about 50 reversals. We know of only 3, possibly 4. This cannot be the whole story. As to the amplitude, it too has increased by about 50, so the original field would have had to have been a few times $10^{-8}$G.