PROBLEM SET 4. SOLUTIONS

Mean Field Dynamo: (20 pts) In the classical mean field dynamo, the key ingredients are amplification of the mean toroidal field by rotational shear, amplification of the mean meridional, or poloidal, field by the $\alpha$ effect, and dissipation of the field by diffusion. The purpose of this problem is to quantify these effects by looking at solutions of the mean field dynamo equations. For simplicity, use a Cartesian coordinate system with $x$ in the direction of rotation, $y$ in the radial direction, and $z$ parallel to the rotation axis. Model the stretching term $r B \cdot \nabla \Omega$ by $\hat{x} B_y \Gamma$, where $\Gamma$ is a constant. Thus, the mean field equations are

$$\frac{\partial B}{\partial t} = \nabla \times \alpha B + \hat{x} \Gamma B_y + \lambda \nabla^2 B.$$ 

Look for axisymmetric plane wave solutions in which $B \propto \exp(pt + i(k_y y + k_z z))$.

1. Obtain expressions for $B_y$ and $B_z$ in terms of $B_x$. **Solution:** The three vector components of the induction equation are

$$pB_x = i\alpha (k_y B_z - k_z B_y) + \Gamma B_y - \lambda k^2 B_x,$$

$$pB_y = ik_z \alpha B_x - \lambda k^2 B_y,$$

$$pB_z = -ik_y \alpha B_x - \lambda k^2 B_z,$$

where $k^2 \equiv k_y^2 + k_z^2$. Thus, we find

$$\frac{B_y}{B_x} = \frac{ik_z \alpha}{p + \lambda k^2},$$

$$\frac{B_z}{B_x} = -\frac{ik_y \alpha}{p + \lambda k^2}.$$

2. Derive a dispersion relation for $p$. Show that the solutions are complex, with an oscillatory part and a growing or decaying part. Verify that if there were no $\alpha$ effect, the solutions would decay. Explain why rotational shear alone is not enough to sustain the field. **Solution:** Using the previous part to eliminate $B_y$ and $B_z$ from the $\hat{x}$ component of the induction equation gives an equation for $B_x$ alone. Factoring out $B_x$ gives the dispersion relation

$$p = -\lambda k^2 \pm (k^2 \alpha^2 + ik_z \alpha \Gamma)^{1/2}.$$ 

1
It is clear that $p$ is complex. This results from rotational shear; if $\Gamma \equiv 0$, $p$ is real, and there is a positive root if $k < |\alpha|/\lambda$. This shows that at sufficiently short wavelengths, resistive damping dominates. It is also clear that there can be no dynamo if $\alpha \equiv 0$. In that case, there is nothing to sustain the poloidal ($B_y, B_z$) components. They decay resistively, leaving no source of toroidal field ($B_x$).

3. Impose the vertical boundary condition $B_z \equiv 0$ at $z = \pm H$. Assume $\Gamma \gg \alpha k$ (rotational shear dominates the $\alpha$ effect) and solve for the minimum half thickness $H$ for which the field can grow. Explain why this lower limit exists. Solution: Instead of assuming the exponential form $\exp ik_z z$ given in the problem, we assume $B_z$ and $B_x$ depend on $z$ as $\cos \pi z/2H$ and $B_y \propto \sin \pi z/2H$. We can see from Part 2 that when rotational shear dominates, the only effect of $k_y$ is to increase the resistive damping, so we take the limit $k_y \to 0$. The dispersion relation reduces to

$$p = -\lambda k_z^2 \pm (1 + i) \left| \frac{k_z \alpha \Gamma}{2} \right|^{1/2}.$$ 

The value of $k_z$ at which $\text{Re}(p) = 0$ is

$$k_z \equiv k_{\text{min}} = \left( \frac{\alpha \Gamma}{2 \lambda^2} \right)^{1/3}.$$ 

Identifying $k_{\text{min}}$ with $\pi/2H$ gives the minimum half thickness of a disk which can sustain a dynamo as $\pi (\lambda^2/4\alpha \Gamma)^{1/3}$. If the disk is thinner and the magnetic field satisfies periodic boundary conditions on its upper and lower surfaces, resistive damping is strong enough to quench the dynamo.

4. Solve the dispersion relation under the conditions in Part 3. Find the fastest growing mode and the ratios $B_y/B_x$ and $B_z/B_x$ for this mode. Compare your results with what you know about the Galactic magnetic field. Solution: Differentiating $\text{Re}(p)$ with respect to $k_z$ and setting the result to zero gives the wavenumber of maximum growth, $k_z(p_{\text{max}})$, as

$$k_z(p_{\text{max}}) = \left( \frac{\alpha \Gamma}{32 \lambda^2} \right)^{1/3}$$

and the maximum growth rate as

$$p_{\text{max}} = \left( \frac{27 \alpha^2 \Gamma^2 \lambda}{1024} \right)^{1/3}.$$ 

2
Using Part 1 and the approximation $\alpha k_z \ll \Gamma$ gives $B_y/B_x \sim O((k_z \alpha/\Gamma)^{1/2}) \ll 1$ while $B_z/B_x \sim k_y/k_z B_y/B_x$. Since in a thin disk it’s reasonable to expect $k_y/k_z \sim H/R$, $B_z$ is even less than $B_x$. Thus, mean field dynamo theory predicts a field which is mostly azimuthal with a small radial component and an even smaller vertical component. This is consistent with the observed field.