# AC Circuits

## 1) INTRODUCTION

In Chapter 22 we learned some things about how to analyze simple DC Circuits. A DC – or direct current – circuit is a one in which the currents and voltages are constant – *i.e.* time independent. There are many applications for DC circuits, but it is much more common to encounter circuits in which the currents and voltages are time dependent.

In the present section we will learn about AC – or alternating current – circuits. In particular we will focus on situations in which the currents and voltages all vary **sinusoidally**, with the signals alternating between positive and negative:
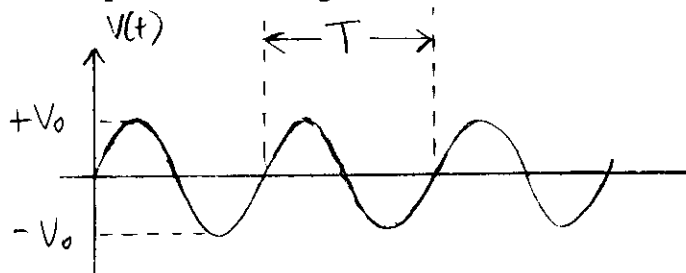


The voltage you get from the wall plugs in your house or apartment is sinusoidal, but there are also many additional applications in which the signals are sine waves.

## 2) SINUSOIDAL FUNCTIONS

Lets define some of the terminology we will be using. In the drawing above the voltage signal varies between $+V_0$ and $-V_0$. We call $V_0$ the amplitude of the signal. The period, $T$, is the time it takes for one complete cycle, and the frequency, $f$, is the number of complete cycles per unit time. As you probably know,

$$f = \frac{1}{T.}$$

The line voltage in your house or apartment has a frequency of 60 Hz (60 cycles/second), and the corresponding period is $T = 16.7\,\text{ms}$.

Here is a simple formula for the voltage $V(t)$ shown above:
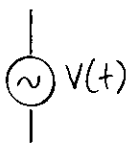
$$V(t) = V_0 \sin \omega t,$$

where $\omega$ is the "angular frequency" of the signal

$$\omega = 2\pi f.$$

It's important to understand that the functions $V_0 \sin \omega t$ and $V_0 \cos \omega t$, are identical except that $\sin \omega t$ is shifted in phase by $\frac{1}{4}$ cycle (or 90°) relative to $\cos \omega t$. One of the things we will learn is that in circuits with inductors and capacitors, and voltages and currents are shifted in phase relative to each other. We will get to that later.
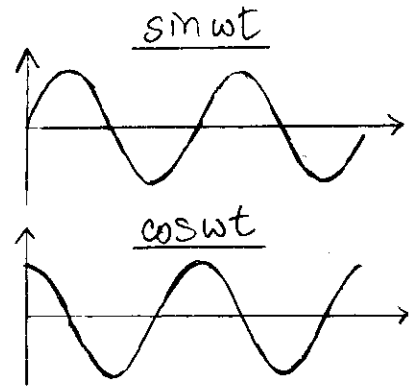
As a first simple example problem lets suppose we apply a sinusoidal voltage to a resistor. If we agree to use the symbol to represent a sinusoidal voltage source (like a battery that makes sine waves), then our circuit would look like this:
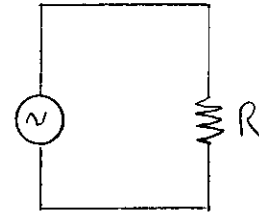
We can find the current in the resistor with Ohm's Law, $V = IR$. This equation applies at each instant in time, so if the voltage is

$$V(t) = V_0 \sin \omega t,$$

the current will be

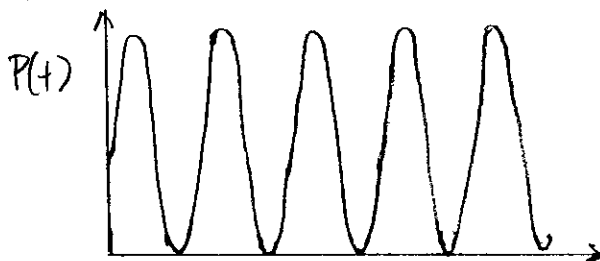$$I(t) = I_0 \sin \omega t,$$

where

$$I_0 = V_0/R.$$

The power dissipated in the resistor is given by $P = I^2 R$. and so we can write

$$P(t) = I_0^2 R \sin^2 \omega t.$$

Notice that the power is time dependent, varying from zero (when $\sin \omega t$ is zero) to $I_0^2 R$ (when $\sin \omega t = +1 \text{ or } -1$). We call this the instantaneous power.

Often we only care about the average power. Since it's the current that depends on time, the

average power $[P]_{\text{ave}}$ is given by

$$[P]_{\text{ave}} = [I^2]_{\text{ave}} R.$$

In general, $[I^2]_{\text{ave}}$ would be called the "mean square" current, and the square root of that quantity would be the "root mean square" (rms) current:

$$I_{\text{rms}} = \sqrt{[I^2]_{\text{ave}}}\,.$$

In terms of the rms current we can write

$$\boxed{[P]_{\text{ave}} = I_{\text{rms}}^2 R.}$$

This is a convenient way to write the power equation because it has the same form as the power law for DC circuits.

It is fairly easy to show that for any sinusoidal current, the average value of $I^2$ is exactly half the peak value:

$$[I^2]_{\text{ave}} = \frac{1}{2} I_0^2,$$

and so we have

$$I_{\text{rms}} = \frac{1}{\sqrt{2}} I_0.$$

We can also define the rms voltage and once again, for any sinusoidal voltage, it will turn out that

$$V_{\text{rms}} = \frac{1}{\sqrt{2}} V_0.$$

Once again, the usefullness of these definitions is that if we choose to work only with the rms quantities and the average power, the relevant equations are $V = IR$ and $P = I^2 R = V^2/R$, exactly as we had earlier for DC circuits.

As you know, the line voltage in our houses is about 115 volts. This number is **not** $V_0$ but actually $V_{\text{rms}}$. If your toaster has a resistance of $50\,\Omega$, the rms current will be $115\,\text{V}/50\,\Omega = 2.3\,\text{A}$ and the corresponding power will be $P = I^2 R = (2.3\,\text{A})^2\,50\,\Omega = 265\,\text{watts}$.

# 3) AC CIRCUITS WITH INDUCTORS

Suppose that we have a circuit that includes an inductor, and that the voltage drop across the inductor sinusoidal: *i.e.*

$$V(t) = V_0 \cos \omega t.$$
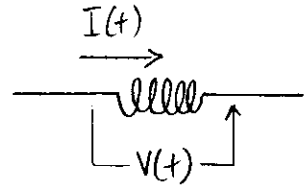
This voltage will give rise to a current, and from the text we know that the voltage and current are related by

$$V(t) = L\frac{d}{dt}I(t).$$

It's easy to see that this equation will be satisfied for a current

$$I(t) = \frac{V_0}{\omega L} \sin \omega t.$$

According to this result, the current will be sinusoidal with an amplitude $I_0 = V_0/\omega L$. If we make the definition

$$\boxed{X_L = \omega L}$$

then the equation relating $I_0$ and $V_0$ has the form of Ohm's Law

$$V_0 = I_0 X_L,$$

where $X_L$ is like the "effective resistance" of the inductor. This quantity, which has units of Ohms, is called the "reactance" of the inductor.

For a regular resistor the current is exactly in phase with the applied voltage, but for the inductor we see that the current is shifted in phase relative to the voltage. As shown in the picture, the current trails the voltage by by 90°. The result is completely sensible, since as we know, the current must increase whenever the applied voltage is positive.

One of the important points here is that the reactance of the inductor depends on the frequency of the AC signal. As the frequency is increased, the reactance increases, which means that the current in the inductor will be smaller.
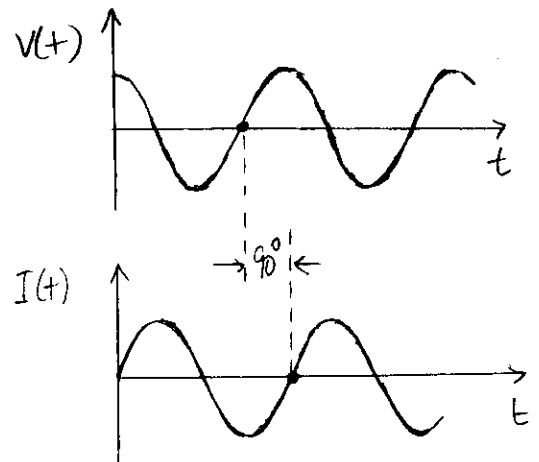
## 4) AC CIRCUITS WITH CAPACITORS

Next lets consider circuits that include capacitors. As in the previous section lets suppose that the voltage across the capacitor is given by the formula

$$V(t) = V_0 \cos \omega t.$$
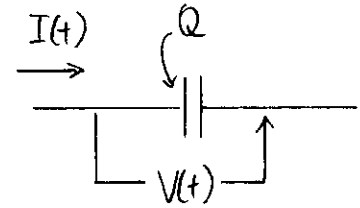
We can find the corresponding current by remembering that the voltage is related to the amount of charge on the capacitor

$$Q(t) = CV(t),$$

and by noticing that positive current flow results in increasing $Q$,

$$\Delta Q = I(t)\Delta t \qquad \text{or} \qquad I = \frac{dQ}{dt}.$$

By taking derivatives of our starting equations we obtain

$$I(t) = \frac{d}{dt}Q(t) = C\frac{d}{dt}V(t) = -\omega C V_0 \sin \omega t.$$

From the last step we see that the current is sinusoidal, and that the peak current and peak voltage are related by

$$V_0 = I_0 X_C$$

where

$$\boxed{X_C = \frac{1}{\omega C}.}$$

As in the inductor example, the quantity $X_C$, which we call the reactance of the capacitor, plays the role of the effective resistance.

For inductors we found that the current trails the voltage by 90°, while in the present case the conclusion (from the equations shown above) is that the current **leads** the voltage by 90°. Once again the result is sensible, since we know that the charge (and therefore the voltage across the capacitor) must increase whenever the current is positive.

Finally we note that, once again, the reactance depends on the frequency of the applied voltage. In this case, increasing the the frequency leads to a smaller reactance and therefore a greater current.
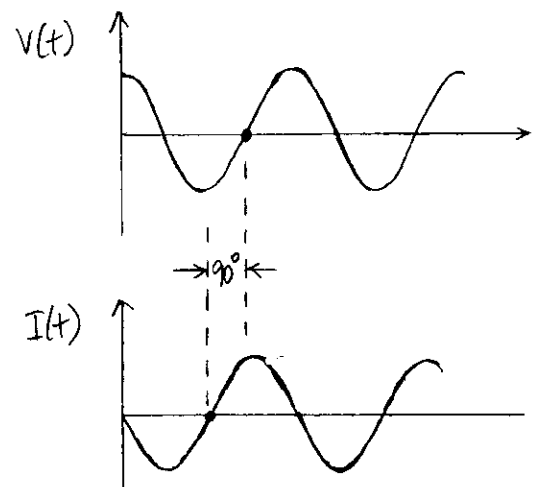
# 5) AN OSCILLATING LC CIRCUIT

The circuit shown at the right is an example of a simple circuit in which the currents and voltages are naturally sinusoidal. Suppose that the capacitor is initially charged to a voltage $V_0$ and that at some time the switch is closed.
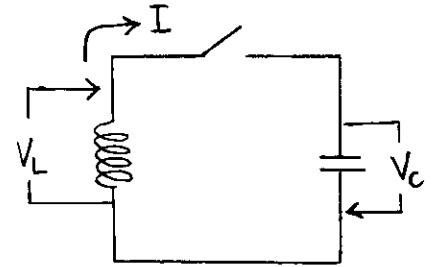
To analyze the circuit lets decide that clockwise currents will be considered to be positive. Following our usual procedure, $V_L$ and $V_C$ stand for the voltage **drops** across the inductor and the capacitor in the direction of positive current as indicated in the drawing. From Kirchhoff's voltage rule we have

$$V_L(t) + V_C(t) = 0,$$

and, in addition, we know that $V_L = L\frac{dI}{dt}$ and that $V_C = \frac{Q}{C}$, so we have

$$L\frac{dI}{dt} + \frac{1}{C}Q = 0.$$

We now take the derivative of this equation and replace $\frac{dQ}{dt}$ by $I$ to obtain

$$L\frac{d^2}{dt^2}I(t) + \frac{1}{C}I(t) = 0$$

or

$$\frac{d^2}{dt^2}I(t) = -\frac{1}{LC}I(t).$$

We should all be able to recognize that this equation is satisfied provided that the current is sinusoidal ($\sin\omega_0 t$ or $\cos\omega_0 t$) and that the angular frequency is

$$\boxed{\omega_0 = \frac{1}{\sqrt{LC}}.}$$

The full solution to our problem, assuming the switch is closed at $t = 0$, would be

$$V_C(t) = -V_L(t) = V_0\cos\omega_0 t \quad \text{and} \quad I(t) = -\frac{V_0}{\omega_0 C}\sin\omega_0 t.$$

The LC circuit is analogous in many respects to the oscillations of a mass on a spring. That system also has a natural frequency, $\omega = \sqrt{k/m}$, and as the mass oscillates there is exchange of

energy, from kinetic to potential, back to kinetic and so on. If the mass/spring system has no friction to dissipate energy, the oscillations will continue forever. In the LC circuit the energy is initially stored in the capacitor, and is then transferred to the inductor, back to the capacitor and so on, and the oscillations continue indefinitely as long as there is no resistance to dissipate the energy.

## 6) SERIES LRC CIRCUITS

As a next example lets consider the series "LRC" circuit shown at the right. Notice that since the elements are connected in series, each of the three elements will have the same current $I(t)$. The voltage we have applied is sinusoidal, and so we know that the current will also be sinusoidal. Finally we know (from Kirchhoff's voltage rule) that the applied voltage from the AC source must equal the sum of the voltage drops across the individual circuit elements

$$V_S(t) = V_R(t) + V_C(t) + V_L(t).$$

The main complication of using this equation is that while the $V_R$, $V_C$ and $V_L$ are all sinewaves, the three voltages are **not in phase** with each other.

Using our results from the sections 3) and 4) above we can learn something about the **amplitude** of each voltage. If $I_0$ is the amplitude of the current, then

$$V_{R,0} = I_0 R \qquad V_{C,0} = I_0 X_C \qquad V_{L,0} = I_0 X_L$$

We also know what the phase relations should be. $V_R(t)$ should be in phase with $I(t)$, $V_C(t)$ should trail $I(t)$ by $90°$ and $V_L(t)$ should lead $I(t)$ by $90°$. Notice that at any given instant, $V_C(t)$ and $V_L(t)$ actually cancell rather than add.
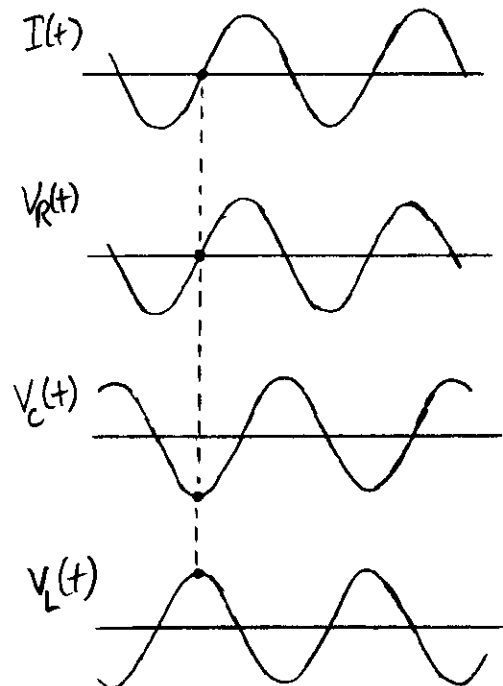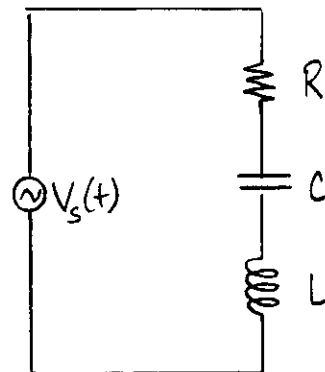
So the problem we have is to add three voltages that are not in phase. We could try to do this with trig formulas, but it's much easier to make a simple geometrical picture. The idea is to make use of the fact that the functions $\sin\theta$ and $\cos\theta$ have simple meanings for right triangles. In

the picture below, $A_0 \cos\theta$ is just the **horizontal component** of a line of length $A_0$. We call the arrow in the drawing a "phasor". To represent the function $A_0 \cos\omega t$ we need to imagine that the phasor rotates counterclockwise (the direction of increasing $\theta$) at angular velocity $\omega$ starting from $\theta = 0$ at time $t = 0$. The horizontal component of the phasor then traces out the cosine function. The function $A_0 \sin\omega t$ can be represented in the same way except that in this case we start at $t = 0$ with the phasor pointing downward.
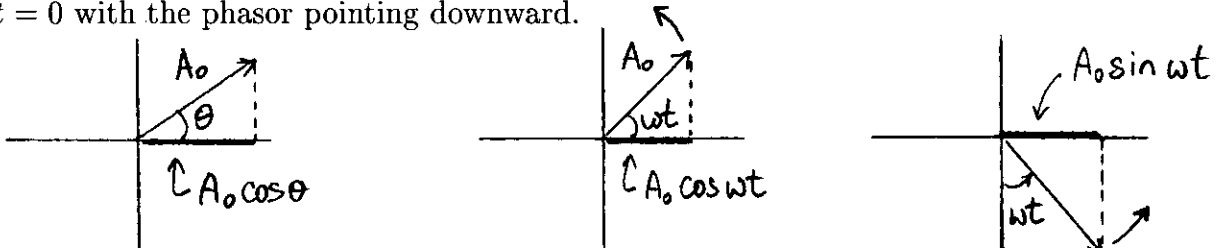
Now lets make a phasor diagram to represent the current and voltages in our LRC circuit. We will have 4 phasors in all, one for the current, one each for $V_R$, $V_C$ and $V_L$. Start by choosing some arbitrary direction for the current (any direction is OK, since different directions just correspond to different times). Once the direction of $I$ is chosen we know how to find the correct directions for the remaining phasors. We just need to remember that $V_R$ is in phase with $I$, $V_L$ leads $I$ by 90° and $V_C$ trails $I$ by 90°.
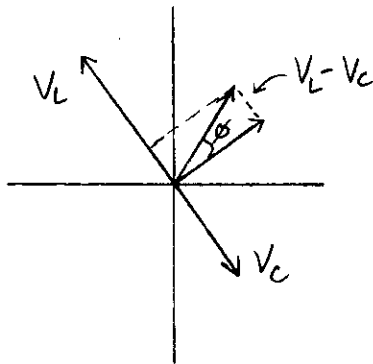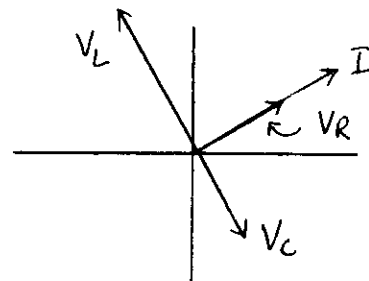
The usefulness of the drawing is that is shows us how to add the three voltages. We just combine the three phasors adding them in the same way that we would add vectors. Recalling that $V_R + V_C + V_L$ is supposed to be $V_S$ we easily find (from the picture) that

$$V_{S,0} = \left[V_{R,0}^2 + (V_{L,0} - V_{C,0})^2\right]^{\frac{1}{2}}.$$

Using the formulas for $V_{R,0}$ etc. from the previous page we can write our result in the form of Ohm's Law

$$V_{S,0} = I_0 Z,$$

where

$$Z = \left[R^2 + (X_L - X_C)^2\right]^{\frac{1}{2}}.$$

The quantity $Z$, which is called the "impedance" of the circuit, can be thought of as the effective resistance of the LRC combination.

Our phasor diagram also provides information about the phase of the current relative to the applied voltage. From the construction above we see that the applied voltage leads the current by an angle $\phi$ given by

$$\tan \phi = \frac{X_L - X_C}{R}.$$

## 7) RESONANCE BEHAVIOR

One of the interesting features of LRC circuits is that the response of the circuit can change dramatically if the frequency of the applied voltage is "tuned" to the "resonant frequency" of the circuit. To understand the resonant behavior we write the impedance of the circuit in the form

$$Z = \left[ R^2 + \left( \omega L - \frac{1}{\omega C} \right)^2 \right]^{\frac{1}{2}}.$$

The resonant frequency is the frequency at which $X_C$ and $X_L$ are equal: *i.e* when

$$\omega = \omega_0 = \frac{1}{\sqrt{LC}}.$$

You may recall that this is just the natural frequency of the LC circuit we discussed earlier.

Right at the resonant frequency, the impedance of the circuit reduces to $R$ and so if $R$ is small we can get relatively large currents, even if the inductor and the capacitor individually have very large impedances. But as soon as we go away from the resonant frequency (say by a few percent) the cancellation between $V_L$ and $V_C$ is no longer complete, and the amplitude of the resulting current is greatly reduced.

Many physical systems will show resonance behavior basically like that of an LRC circuit. If the system has a natural oscillation frequency and there is no mechanism to dissipate energy, large amplitude oscillations occur when the system is driven at the natural frequency. Resonances are seen in mechanical systems, electrical systems and also in atoms and nuclei. For example MRI scans (Magnetic Resonance Imaging) make use of a technique for detecting nuclei of hydrogen atoms in our bodies by stimulating the nuclei at a frequency equal to the precession frequency of the nuclear magnetic dipole moments in the magnetic field of the scanner. The inventers of this technique just won the Nobel Prize.