EXAM I SOLUTIONS

1)(a) $f(v) = -bv^{1/2} \Rightarrow \quad m\ddot{x} = m\frac{dv}{dt} = -bv^{\frac{1}{2}}$

Separate variables and integrate

$$\int_{v_0}^{v} \frac{dv}{v^{1/2}} = -\frac{b}{m}\int_0^t dt$$

$$2\sqrt{v}\Big|_{v_0}^{v} = -\frac{b}{m}t$$

$$\sqrt{v} - \sqrt{v_0} = -\frac{b}{2m}t \quad \Rightarrow \sqrt{v} = \sqrt{v_0} - \frac{b}{2m}t$$

$$\boxed{v = \left[\sqrt{v_0} - \frac{b}{2m}t\right]^2}$$

(b) $v = 0$ when $\qquad \sqrt{v_0} - \frac{b}{2m}t = 0$

$\Rightarrow$

$$\boxed{t_f = \frac{2m}{b}\sqrt{v_0}}$$

(c) $v = \frac{dx}{dt} = \left[\sqrt{v_0} - \frac{b}{2m}t\right]^2 = v_0 - 2\sqrt{v_0}\left(\frac{b}{2m}t\right) + \left(\frac{b}{2m}\right)^2 t^2$

$$\int_{x_0}^{x} dx = \int_0^t \left[v_0 - 2\sqrt{v_0}\left(\frac{b}{2m}\right)t + \left(\frac{b}{2m}\right)^2 t^2\right] dt$$

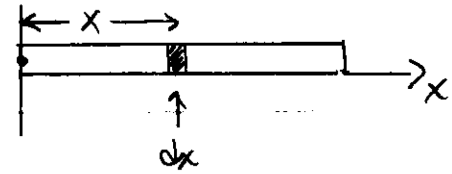
$$\underline{x - x_0 = v_0 t - \sqrt{v_0}\left(\frac{b}{2m}\right)t^2 + \frac{1}{3}\left(\frac{b}{2m}\right)^2 t^3}$$

Distance travelled at time t_f is

$$S = \cancel{v_0\left(\frac{2m}{b}\right)\sqrt{v_0} - \sqrt{v_0}\left(\frac{b}{2m}\right)\left(\frac{2m}{b}\right)^2 v_0} + \frac{1}{3}\left(\frac{b}{2m}\right)^2\left(\frac{2m}{b}\right)^3 v_0^{3/2}$$

$$\boxed{S = \frac{2}{3}\frac{m}{b}v_0^{3/2}}$$

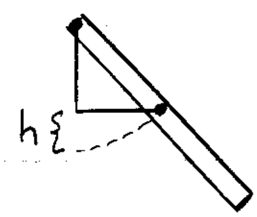
2) (a) $I = \sum_\alpha m_\alpha \rho_\alpha^2$. Divide the stick into segments of length dx. The mass m_α of the segment is $(dx)\cdot$(mass per unit length) where mass per unit length $= M/L$



$$I = \int_0^L (x^2)\left(\frac{M}{L}\right)dx = \frac{1}{3}\frac{M}{L}x^3\Big|_0^L = \boxed{\frac{1}{2}ML^2}$$

(b) The kinetic energy is $T = \frac{1}{2}I\omega^2 = \frac{1}{2}I\dot{\theta}^2$

[to see that notice that the velocity of mass at a distance ρ_α from the pivot is $v_\alpha = \rho_\alpha \omega \Rightarrow T_\alpha = \frac{1}{2}m_\alpha v_\alpha^2 = \frac{1}{2}m_\alpha \rho_\alpha^2 \omega^2$ $\Rightarrow T = \frac{1}{2}I\omega^2$] To find U, consider all the mass to be at the C.M. $\Rightarrow$



$$h = (L/2)(1-\cos\theta) \Rightarrow U = Mg\frac{L}{2}(1-\cos\theta)$$

$$E = T + U = \frac{1}{2}I\dot{\theta}^2 + Mg\frac{L}{2}(1-\cos\theta)$$

$$\boxed{E = \left(\frac{1}{6}\right)ML^2\dot{\theta}^2 + \frac{L}{2}Mg(1-\cos\theta)}$$

(c) E is constant so $\frac{dE}{dt} = 0$

$$\frac{d}{dt}E = \left(\frac{1}{6}\right)\cancel{M}L^2(2)\cancel{\dot{\theta}}\ddot{\theta} + \frac{1}{2}\cancel{M}gL\sin\theta\,\cancel{\dot{\theta}} = 0$$

$$\boxed{\ddot{\theta} = -\frac{3}{2}\frac{g}{L}\sin\theta}$$

3) A force is conservative if $\vec{\nabla} \times \vec{F} = 0$

(a) $\nabla \times F = \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{bmatrix} = \hat{x}\left[\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right] + \hat{y}\left[\frac{\partial}{\partial z}F_x - \frac{\partial}{\partial x}F_z\right]$

$+ \hat{z}\left[\frac{\partial}{\partial x}F_y - \frac{\partial}{\partial y}F_x\right]$

$= \hat{x}\left[9abxz^2 - 9abxz^2\right] + \hat{y}\left[9abyz^2 - 9abyz^2\right]$

$+ \hat{z}\left[(3abz^3 - 8bx^3) - (3abz^3 - 8bx^3)\right] = 0$

So Force (a) is conservative.

(b) $\vec{\nabla} \times \vec{F} = \hat{x}\left[6abxz^2 - 6abxz^2\right] + \hat{y}\left[6abyz^2 - 6abyz^2\right]$

$+ \hat{z}\left[(2abz^3 - 32bx^3) - (2abz^3 - 2bx^3)\right] = -30bx^3\,\hat{z} \neq 0$

$\Rightarrow$ (b) is not conservative.

To find the potential we use $U = -\int_{\vec{r}_s}^{\vec{r}} \vec{F} \cdot d\vec{s}$ where the integral is along a line from some reference point to an arbitrary point $\vec{r}$ at which we are finding U. Choose the reference point to be $\vec{r}_s = (0,0,0)$ = the origin. From there I will integrate out along the x-axis ($x \to 0$ to x with $y = z = 0$), then along y ($y \to 0$ to y with $x = x, z = 0$) and then along z ($z \to 0$ to z with $x = x, y = y$).

$-U = \int_0^x F_x(x,0,0)\,dx + \int_0^y F_y(x,y,0)\,dy + \int_0^z F_z(x,y,z)\,dz$

$= 0 + \int_0^y \left[0 - 2bx^4\right]dy + \int_0^z (9abxyz^2)\,dz$

$= -2bx^4y + 3abxyz^3$ $\qquad \boxed{U = -3abxyz^3 + 2bx^4y}$