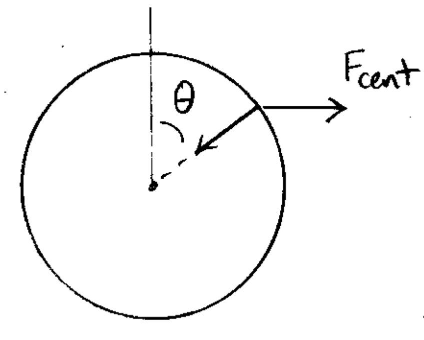
EXAM 2 SOLUTIONS



1) The effective gravitational force is $m\vec{g}_0$ plus the <u>outward</u> centrifugal force

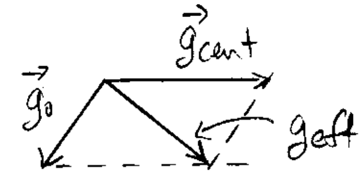
$$\vec{F} = m\vec{g}_0 + m\Omega^2\rho\hat{\rho}$$

$\Rightarrow$

$$\vec{g}_{eff} = \vec{g}_0 + \Omega^2\rho\hat{\rho} \qquad ; \ \Omega = \frac{2\pi}{3600s} = 0.00175\,rad/s.$$

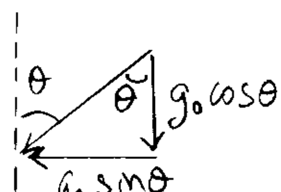
$\Rightarrow$

$$\Omega^2\rho = \Omega^2 R_E \sin\theta = 14.26\ ^m/_{s^2}$$



Now add the two vectors. $\longrightarrow$

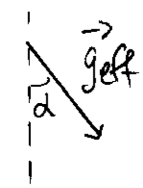
I will resolve $\vec{g}_0$ into $\hat{z}$ and $\hat{\rho}$ components



$$g_z = -g_0\cos\theta \quad , \quad g_\rho = -g_0\sin\theta$$

$$g_{eff} = (-g_0\cos\theta\,;\ \Omega^2\rho - g_0\sin\theta) = (-6.68\,;\ 7.09)$$

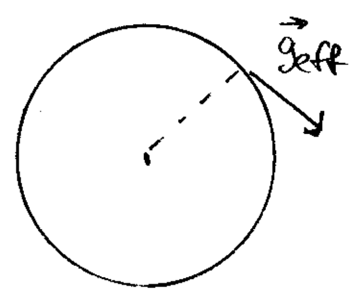
$$g_{eff} = \left[(7.09)^2 + (6.68)^2\right]^{\frac{1}{2}} = \boxed{9.74\ ^m/_{s^2}}$$



and

$$\tan\alpha = \frac{7.09}{6.68} \Rightarrow \alpha = 46.7$$

$\measuredangle$ between g_0 and g_{eff} $= \theta + \alpha = 47^\circ + 46.7^\circ = \boxed{93.7^\circ} \Rightarrow$ sideways



2) (a) $T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{r}^2 + (r\dot{\phi})^2)$

$$\mathcal{L} = T - U = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 + \frac{k}{r^{3/2}}$$

$$\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = mr^2\dot{\phi} \qquad \frac{d}{dt}(mr^2\dot{\phi}) = 0 \quad \Rightarrow \quad \boxed{mr^2\dot{\phi} = \ell = \text{constant}} \quad \dot{\phi} = \frac{\ell}{mr^2}$$

(b) $\frac{\partial\mathcal{L}}{\partial t} = 0$ so $H = \text{constant} = \sum p_k\dot{q}_k - \mathcal{L}$

$$\sum_k p_k\dot{q}_k = (m\dot{r})(\dot{r}) + (mr^2\dot{\phi})(\dot{\phi}) = m\dot{r}^2 + mr^2\dot{\phi}^2$$

so $H = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 - k/r^{3/2}$

$$= \frac{1}{2}m\dot{r}^2 + \frac{\ell^2}{2mr^2} - k/r^{3/2} = \frac{1}{2}m\dot{r}^2 + U_{eff}$$

$$\boxed{U_{eff} = \frac{\ell^2}{2mr^2} - \frac{k}{r^{3/2}}}$$

Equilibrium occurs when $\frac{dU}{dr} = 0$

$$\frac{dU}{dr} = -2\left(\frac{\ell^2}{2mr^3}\right) - \left(-\frac{3}{2}\right)\frac{k}{r^{5/2}} = 0 \Rightarrow \frac{3}{2}\frac{k}{r^{5/2}} = \frac{\ell^2}{mr^3}$$

$$r^{\frac{1}{2}} = \frac{2}{3}\frac{\ell^2}{km} \qquad \boxed{r_0 = \left(\frac{2}{3}\frac{\ell^2}{km}\right)^2}$$

(c) The orbits are stable if U'' = positive.

$$\frac{d^2U}{dr^2} = +3\frac{\ell^2}{mr^4} + \left(\frac{3}{2}\right)\left(-\frac{5}{2}\right)\frac{k}{r^{7/2}} = \frac{1}{r^4}\left[3\frac{\ell^2}{m} - \frac{15}{4}kr^{\frac{1}{2}}\right]$$

$$\left.\frac{d^2U}{dr^2}\right|_{r_0} = \frac{1}{r_0^4}\left[3\frac{\ell^2}{m} - \frac{15}{4}k\frac{2}{3}\frac{\ell^2}{km}\right] = \frac{1}{r_0^4}\left[3 - \frac{5}{2}\right]\frac{\ell^2}{m}$$

U'' is positive so the orbits are <u>stable</u>.

3) (a) $T_2 = \frac{1}{2}m_2\dot{s}^2$

For m_1, $x = s + \ell\sin\theta \qquad \dot{x} = \dot{s} + \ell\cos\theta\,\dot{\theta}$

$y = \ell(1-\cos\theta) \qquad \dot{y} = \ell\sin\theta\,\dot{\theta}$

$$T_1 = \frac{1}{2}m_1\left[\dot{x}^2 + \dot{y}^2\right] = \frac{1}{2}m_1\left[\dot{s}^2 + 2\ell\cos\theta\,\dot{s}\dot{\theta} + \ell^2\cos^2\theta\,\dot{\theta}^2 + \ell^2\sin^2\theta\,\dot{\theta}^2\right]$$

$$T = T_1 + T_2 = \frac{1}{2}(m_1+m_2)\dot{s}^2 + m_1\ell\cos\theta\,\dot{s}\dot{\theta} + \frac{1}{2}m_1\ell^2\dot{\theta}^2$$

$U = mgy$

$$\boxed{\mathcal{L} = \frac{1}{2}(m_1+m_2)\dot{s}^2 + m_1\ell\cos\theta\,\dot{s}\dot{\theta} + \frac{1}{2}m_1\ell^2\dot{\theta}^2 - mg\ell(1-\cos\theta)}$$

(b) 1) $\frac{\partial \mathcal{L}}{\partial s} = 0$ so, $\frac{\partial \mathcal{L}}{\partial \dot{s}}$ = constant

$$\frac{\partial \mathcal{L}}{\partial \dot{s}} = \boxed{(m_1+m_2)\dot{s} + m_1 \ell \cos\theta\, \dot{\theta} = C_1}$$

This is

$$m_1(\dot{s} + \ell\cos\theta\,\dot{\theta}) + m_2\dot{s} = m_1\dot{x}_1 + m_2\dot{s} = x \text{ component of } \vec{P}_{TOT}$$

So the <u>horizontal component of the total momentum is conserved</u>

2) $\frac{\partial \mathcal{L}}{\partial t} = 0 \Rightarrow H = \text{constant} = \sum_k p_k \dot{q}_k - \mathcal{L}$

$$\begin{aligned}\sum_k p_k \dot{q}_k &= [(m_1+m_2)\dot{s} + m_1\ell\cos\theta\,\dot{\theta}]\dot{s} + [m_1\ell\cos\theta\,\dot{s} + m_1\ell^2\dot{\theta}]\dot{\theta} \\ &= (m_1+m_2)\dot{s}^2 + 2m_1\ell\cos\theta\,\dot{s}\dot{\theta} + m\ell^2\dot{\theta}^2 \\ &= 2T\end{aligned}$$

So

$$H = \sum_k p_k \dot{q}_k - \mathcal{L} = \boxed{T + U = \underline{\text{total energy}} = \text{constant}} = C_2$$

$$= \tfrac{1}{2}(m_1+m_2)\dot{s}^2 + m_1\ell\cos\theta\,\dot{s}\dot{\theta} + \tfrac{1}{2}m_1\ell^2\dot{\theta}^2 + mg\ell(1-\cos\theta)$$

(c) $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{d}{dt}[m_1\ell\cos\theta\,\dot{s} + m_1\ell^2\dot{\theta}] = \frac{\partial \mathcal{L}}{\partial \theta} = -m_1\ell\sin\theta\,\dot{s}\dot{\theta} - mg\ell\sin\theta$

$$\underline{m_1\ell\cos\theta\,\ddot{s} + \cancel{m_1\ell(-\sin\theta\,\dot{\theta})\dot{s}} + m_1\ell^2\ddot{\theta} = \cancel{-m_1\ell\sin\theta\,\dot{s}\dot{\theta}} - m_1 g\ell\sin\theta}$$

From above, we can eliminate $\ddot{s}$ by taking $\frac{d}{dt}$ of:

$$(m_1+m_2)\dot{s} + m_1\ell\cos\theta\,\dot{\theta} = C$$

$\Rightarrow$

$$(m_1+m_2)\ddot{s} + m_1\ell\cos\theta\,\ddot{\theta} - m_1\ell\sin\theta\,\dot{\theta}^2 = 0$$

$$\ddot{s} = \left(\frac{m_1}{m_1+m_2}\right)\ell\,[\sin\theta\,\dot{\theta}^2 - \cos\theta\,\ddot{\theta}]$$

so the equation of motion is

$$\cancel{m_1}\cancel{l}\cos\theta\left(\frac{m_1}{m_1+m_2}\right)l\left[\sin\theta\,\dot{\theta}^2-\cos\theta\,\ddot{\theta}\right]+\cancel{m_1}\,l^{\cancel{2}}\,\ddot{\theta}=-\cancel{m_1}\,g\,\cancel{l}\sin\theta$$

$$\boxed{\left[1-\left(\frac{m_1}{m_1+m_2}\right)\cos^2\theta\right]\ddot{\theta}+\left(\frac{m_1}{m_1+m_2}\right)\sin\theta\cos\theta\,\dot{\theta}^2=-\left(\frac{g}{l}\right)\sin\theta}$$