FINAL REVIEW

CHAPTER 10: Rigid Body Rotations

- For any system of particles $\frac{d}{dt}\vec{L} = \vec{\Gamma}^{EXT}$ for any inertial reference frame. O.
- The angular momentum can be divided into two parts
$$\vec{L} = \vec{L}_{orbit} + \vec{L}_{spin}$$
where $\vec{L}_{orbit}$ is the $\vec{L}$ associated with motion of C.M. about O. and $\vec{L}_{spin}$ is the angular momentum of rotation about C.M.
- We demonstrated that
$$\frac{d}{dt}\vec{L}_{spin} = \vec{\Gamma}^{EXT} \text{ (about the C.M.)}$$
So for a rotating rigid body in "free fall" (gravity only)
$$\vec{L}_{spin} = \text{constant}$$
- $\vec{\omega} \equiv$ instantaneous angular velocity (direction of $\vec{\omega}$ is the instantaneous rotation axis).
- $\vec{L} = \overleftrightarrow{I} \cdot \vec{\omega}$ $\quad I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$ $\quad I_{xx} = \sum_\alpha m_\alpha (y_\alpha^2 + z_\alpha^2)$, $\quad I_{xy} = -\sum_\alpha m_\alpha x_\alpha y_\alpha$

or
$$\overleftrightarrow{I} = \left[\sum_\alpha m_\alpha \vec{r}_\alpha \cdot \vec{r}_\alpha\right] \overleftrightarrow{1} - \sum_\alpha m_\alpha \vec{r}_\alpha \vec{r}_\alpha$$
- A principal axis is any axis for which $\vec{L} \parallel \vec{\omega}$
- It's possible to find 3 mutually $\perp$ principal axes for any rigid body. If we choose these as our coordinate axes then $\overleftrightarrow{I}$ will be diagonal: $\quad I = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$
- We can find the principle axes and moments of inertia by solving
$$\begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \lambda \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \Rightarrow \text{Det} \begin{bmatrix} I_{xx}-\lambda & I_{xy} & I_{xz} \\ I_{yx} & I_{yy}-\lambda & I_{yz} \\ I_{zx} & I_{zy} & I_{zz}-\lambda \end{bmatrix} = 0$$

- The rotational Kinetic Energy is $T = \frac{1}{2}\vec{\omega}\cdot\vec{L}$. $\Rightarrow$ using principal axes $T = \frac{1}{2}(\lambda_1\omega_1^2 + \lambda_2\omega_2^2 + \lambda_3\omega_3^2)$
- Euler's Equations

$$\Gamma_1 = I_1\dot{\omega}_1 + (\lambda_3 - \lambda_2)\omega_3\omega_2$$
$$\Gamma_2 = I_2\dot{\omega}_2 + (\lambda_1 - \lambda_3)\omega_1\omega_3$$
$$\Gamma_3 = I_3\dot{\omega}_3 + (\lambda_2 - \lambda_1)\omega_2\omega_1$$

CHAPTER 11: Coupled Oscillators

- Two coupled oscillators will give equations of the form

$$m_{11}\ddot{x}_1 + m_{12}\ddot{x}_2 = -(k_{11}x_1 + k_{12}x_2)$$
$$m_{21}\ddot{x}_1 + m_{22}\ddot{x}_2 = -(k_{21}x_1 + k_{22}x_2)$$

- Linear equations are solved by trying

$$x_1(t) = a_1 e^{i\omega t} \qquad x_2(t) = a_2 e^{i\omega t}$$

Substitute and solve for ω. We get a quadratic equation for ω^2 $\Rightarrow$ $a\omega^4 + b\omega^2 + c = 0$ $\Rightarrow$ 2 roots ω_1^2, ω_2^2 which are the normal mode frequencies.

- Each normal mode has it's own ratio of amplitudes, $r = a_2/a_1$, which can be found from the equations of motion. The general solution for the motion is

$$x_1(t) = A_1\cos(\omega_1 t - \delta_1) + A_2\cos(\omega_2 t - \delta_2)$$
$$x_2(t) = A_1 r_1\cos(\omega_1 t - \delta_1) + A_2 r_2\cos(\omega_2 t - \delta_2)$$

$\hookrightarrow$ first normal mode $\qquad$ $\hookrightarrow$ second normal mode

- In matrix form, the equation of motion is $M\ddot{\underset{\sim}{x}} = -K\underset{\sim}{x}$. If there are n coordinates M and K are $n\times n$ matricees and $\underset{\sim}{x}$ is a $1\times n$ column vector. Writing

$$x_i = a_i e^{i\omega t}$$

gives an "eigenvalue equation"

$$K\underset{\sim}{a} = \omega^2 M\underset{\sim}{a}$$

and the common mode frequencies are obtained from the secular equation

$$\text{Det}\left[K-\omega^2 M\right]=0.$$

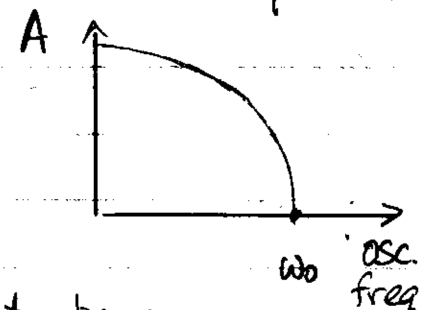
If there are n coordinates there will be n common modes, each with its own frequency ω_i.

CHAPTER 12: Chaos

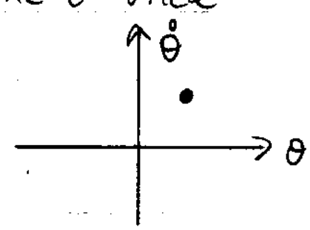
- Damped, Driven Pendulum

$$\ddot{\theta} + 2\beta\dot{\theta} + \omega_0^2 \sin\theta = \gamma\,\omega_0^2 \cos\omega t$$

γ = driving force ; ω = driving frequency ; $\omega_0 = \sqrt{\frac{g}{\ell}}$ = natural frequency for small-amplitude oscillations. For γ and $\beta = 0$ the natural oscillation frequency depends on amplitude:



- Small γ: Steady-state motion is simple and periodic - at the frequency of the driving force: $T_{motion} = T_{drive}$. Transients die away $\Rightarrow$ S.S. motion is independent of starting conditions.
- Larger γ: Motion is still periodic but conditions can be found that permit two or more distinct steady state motions (e.g. large amplitude vs small amplitude or motions not symmetric about $\theta=0 \Rightarrow \theta_2(t) = -\theta_1(t+T/2)$). Transients still die away, but motion may attract to solution 1 or solution 2 depending on initial conditions
- Poincaré Sections: Plot "state of the system" $\Rightarrow \theta$ and $\dot{\theta}$ once per cycle. Periodic motion $\Rightarrow$ one repeating point. If we start at some arbitrary point, $\theta, \dot{\theta}$ values from subsequent cycles attract to the stable points.



- <u>Mapping Functions:</u> Mathematical rule that predicts the state at cycle n+1 given the state at cycle n

$$\theta_{n+1} = f(\theta_n, \dot\theta_n) \qquad \dot\theta_{n+1} = g(\theta_n, \dot\theta_n)$$

Periodic behavior results from <u>fixed points</u> of the mapping functions:

$$f(\theta^*, \dot\theta^*) = \theta^* \qquad g(\theta^*, \dot\theta^*) = \dot\theta^*$$

Fixed points can be stable (attractors) or unstable.

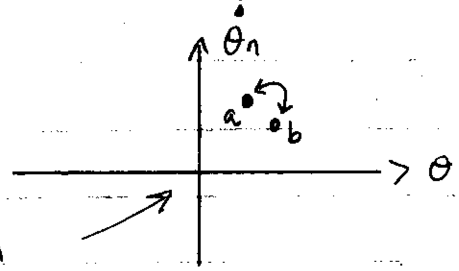
- <u>1-Parameter Mapping Functions:</u> $x_{n+1} = F(x_n)$

Fixed points: $x^* = F(x^*)$.

Fixed points are stable (attractors) if $|F'(x^*)| < 1$

Example: $F(x) = rx(1-x)$

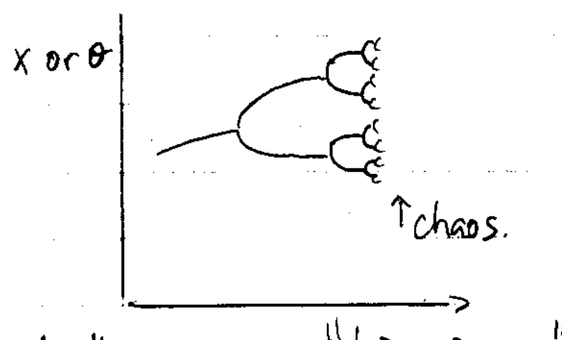
- <u>Transition to Chaos:</u>



i) <u>Mapping functions "bifurcate":</u> e.g. $F(x_a) = x_b$; $F(x_b) = x_a$. Pendulum bifurcates between 2 system points ⇒ <u>Period doubling</u> $T_{motion} = 2T_{drive}$.

ii) This is followed by a <u>sequence of further period doublings coming at closer + closer intervals.</u>

iii) The <u>sequence of doubling points</u> has a <u>finite limit</u> ⇒ <u>chaos with motion that never repeats</u>.



- <u>Chaotic Motion</u>:

Tiny changes in initial conditions eventually grow into large differences.

Motion is complicated but not random. Motion still attracts to some kind of complex structure.